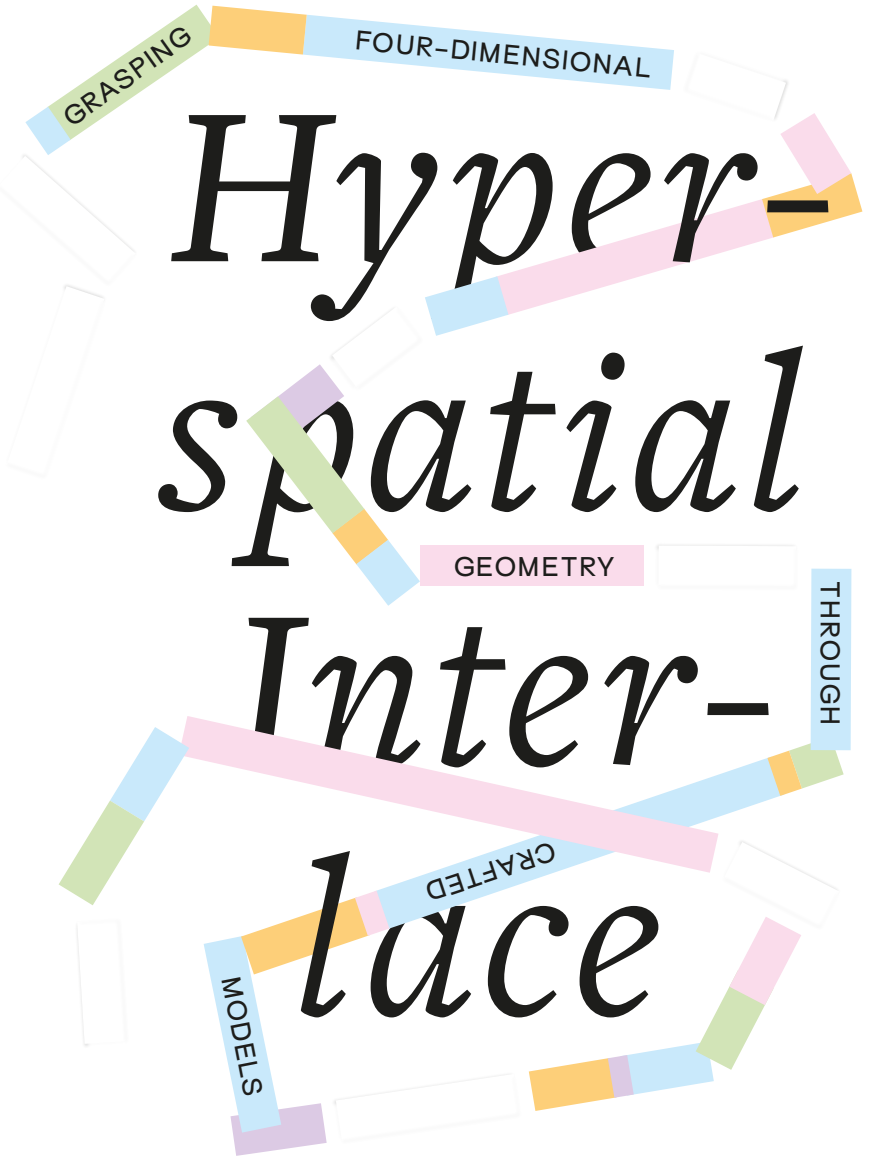

TANELI LUOTONIEMI

Aalto University

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GRASPING FOUR-DIMENSIONAL

Hyper-spatial Inter-lace

GEOMETRY

THROUGH

CRAFTED

MODELS

Hyperspatial Interlace
– Grasping Four-dimensional Geometry Through Crafted Models

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ABSTRACT

Although our everyday perception of space tells us that it is composed of three spatial dimensions of length, height and width, it is possible to imagine a space having four spatial directions that are exactly identical with respect to each other, and meet at right angles. This concept of a four-dimensional space has had a profound and lasting effect not only on philosophy, mysticism, mathematics, and theoretical physics, but also on fiction and visual arts.

Because the mathematically described four-dimensional structures are difficult to grasp visually, it is instructive to observe spaces of lower and higher dimensionality in their hierarchical relation. In this context we can see that our challenges to understand and visualize four-dimensional objects are analogous to the difficulties a two-dimensional being, confined to a plane, would have with respect to our three-dimensional space and its shapes. Consequently, just as three-dimensional structures can be drawn, unfolded, sliced, photographed or otherwise projected onto a two-dimensional medium like paper or a computer screen, these graphical techniques can be generalized to acquire the three-dimensional appearances of the four-dimensional hypersolids.

The objective of the artistic research reported here was to craft physical, three-dimensional models that illustrate regular hyperspatial structures through novel visualization methods. I considered kinetic models of the four-dimensional polytopes in particular to be beneficial for understanding these structures. As a

research method I employed experimental crafting with traditional materials and techniques, and the theory guiding these constructions came from the field of descriptive geometry. The cultural references in the history of hyperspace, such as interpretations in spiritualism and science fiction, served as a poetic inspiration in designing the models.

As the results of my research, I present five concepts for physical models, each of which illustrates a hypersolid through a different visualization method. The set of objects consists of a stereographic projection of the hexadecachoron made from brass hoops, perspective models of the tesseract in steel wire and paper, the bitruncated versions of the pentachoron and the tesseract in topological cloth patchworks, a set of stick models depicting the icositetrachoron in a gnomonic projection, and a beadwork 'cavalier projection' of the 3-3 duoprism.

These objects illuminate new connections between arts and mathematics, and serve to enrich the morphological repertory of visual art practice with novel means and meanings. As pedagogical tools, the models pursued offer a hands-on experience of hyperspatial geometry, thus democratizing pure mathematics. In the context of artistic research my work presents itself as an example of an unconventional multidisciplinary methodology.

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A doctoral thesis is always the product of a collaborative effort. Here the importance of allies cannot be exaggerated, as I have had the privilege of relying on invaluable help and guidance from several of my superiors, peers, friends, and loved ones. It is almost overwhelming to think of all the opportunities, support, and trust I have received throughout the whole process, and from so many people.

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INTRO–DUCTION

The project presented here is an artistic research focused on the visual aspects of geometric structures – a profoundly interdisciplinary project positioned between mathematics and art education. The result of the research is a family of crafted objects – prototypes for pedagogical models, portraying various four-dimensional structures through different geometric techniques. Although the scope of the work prevents me from sufficiently introducing the geometry involved, I wish to familiarize the reader with a few basic concepts before giving the report of my research. These aspects are related to spatial dimensions, structures enabled by them, and the methods used to portray them in lower-dimensional spaces. These concepts are described quite informally here, and it is important to note that all of them have considerable mathematical discussion connected to them, which is not possible to treat here. For my non-mathematician readers desiring a popular introduction on the topic, I recommend the works of e.g. Manning[1], Weeks[2], and Rucker[3].

1 Manning 1960.
2 Weeks 2002.
3 Rucker 2014.

BACKGROUND

The disciplines of mathematics and art intersect at many levels. Beside their historical relationship, both disciplines have the power to evoke in their practitioners and audiences experiences described with words like 'beautiful', 'elegant' and 'revealing'. Usually mathematics is thought of as supplying artists with rational tools (e.g. linear perspective or the golden section) when constructing presentations of their more 'humane' content, but there are also artists who get their subject matter from the same spatial phenomena (e.g. symmetries, proportions, mappings, projections and patterns) that are studied by mathematics in the frameworks of geometry and topology.

Yet there is an even stronger connection between the two disciplines. In mathematics, as well as in art, it is perfectly acceptable to investigate things that do not exist in physical or human reality as perceived in the domains of natural sciences and humanities. Like a work of art, a mathematical conception can potentially, one day, increase our understanding of reality, but how well they correspond with the actual world is a test neither of good mathematics nor good art. One of the most intriguing examples of such an imaginative notion is the mathematical and cultural tradition of the fourth spatial dimension.

THE FOURTH SPATIAL DIMENSION

This work begins as a thought experiment, a geometric fantasy. We imagine a space, where instead of our familiar three dimensions of length, height and width, there would be *four spatial dimensions*. The word 'spatial' has to be taken seriously here, as it excludes the possibility of interpreting time as the fourth dimension. Consequently, the concept of four-dimensional space-time discussed in physics falls outside the scope of this study altogether[4]. Just as the first three spatial dimensions are understood as being exactly identical with respect to each other, so the fourth spatial dimension is imagined as being just like the previous three, and meeting them all at right angles. Coming into contact with it for the first time, it may come as a surprise that four-dimensional space is neither an absurd

4 As the pioneer of four-dimensional visualization Tom Banchoff likes to put it, time is *a* fourth dimension, not *the* fourth dimension, and that time should perhaps be called the first dimension, since we need time to do anything.

concept, nor does the reasoning involved lead to logical dead-ends or paradoxes. Instead, research on the geometry of four-dimensional space is a consistent and mathematically rigorous activity.

The investigations on 4-space are made possible by generalizing the geometric principles acquired by studying the more familiar spaces of lower dimensions. This approach assumes the spatial dimensions are set up as an ascending hierarchy. A point is a space with zero dimensions, and a line has a dimension of one. A plane is two-dimensional, whereas three dimensions comprise our ordinary space. In the context of the present discussion work we will refer to it as *3-space* to avoid confusion. The topic of the study at hand, four-dimensional space, is a fanciful but nevertheless logically natural extension of this hierarchy, and is often referred to as *4-space*, or *hyperspace* – terms also used in this work.

From this ascending ladder, we can say the following: spaces of lower dimension are contained in spaces of higher dimensions, and regions of lower dimensionality can bound regions from spaces of higher dimensions. It is important to realize that the fourth spatial dimension, in its geometric context, is conceived as being *exactly similar* with the previous three. Although the fourth spatial dimension seems special to us, it is – in its correct context of 4-space – identical with all the others, and the dimensions can even be interchanged via a rotation.

The number of physical dimensions actually existing – or rather, the absoluteness of four-dimensional space – has been a subject of debate since antiquity. Early proofs of the physical existence of exactly three spatial dimensions were pursued through considering the maximum number of mutually perpendicular lines emanating from a point. Such explanations were already being discussed by Aristotle in his 350 BC cosmological treatise *On the Heavens*[5], as well as by Ptolemy, and later by Descartes, and Leibniz[6]. Although in the 3rd century Diophantus of Alexandria mentions *square-square*[7], *square-cube*, and *cube-cube*,[8] the 16th century mathematician Stifel still speaks of "going beyond the cube" just as if it were as being "against nature", and a

5 Manning 1956, 1.
6 Whitrow 1955.
7 We will see how Diophantus' square-square gets an explicit manifestation as the tesseract, the "triangle-triangle" being the subject matter of another visualization case, the *Prismary*.
8 Manning 1956, 2.

century later John Wallis declared "ungeometrical" higher powers as "less possible than a Chimæra or Centaure" in his *Algebra*[9].

Immanuel Kant was already on the threshold of the hyperspatial revelation when he wondered why there are exactly three dimensions of our observable universe, and was puzzled over mirror images of three-dimensional shapes, such as left and right hands, that are exactly alike in every geometric respect, yet cannot be made to coincide[10].

It was not until the 19th century when this mystery gained its explanation in a mathematical description of the fourth spatial dimension. German mathematician August Ferdinand Möbius introduced rotation around a plane in 1827 – a manipulation mathematically possible in 4-space, which (although he judged "it cannot be thought") indeed turns three-dimensional figures into their mirror images[11]. In 1846 Arthur Cayley talked of four-dimensional space where "planes" would intersect at "half-planes", referring to hyperplanes and ordinary planes, respectively[12]. A year later Augustin-Louis Cauchy discussed the possibility of "analytical points" having coordinates greater than three[13], and Riemann's seminal 1854 paper *On the Hypotheses That Lie at the Foundations of Geometry* discusses the geometrical possibility of manifolds having *n* dimensions. George Salmon mentioned higher-dimensional spaces as a means to solve algebraic problems in 1866[14], and a Canadian-American astronomer Simon Newcomb proved in 1878 that if a closed surface such as a sphere is placed in a space of more than three dimensions, it can be turned inside out without puncturing or tearing.[15] He later illustrated this idea with a lower-dimensional analogy of a rubber annulus having its outer rim glued to a table top, and pulling the inner rim upwards, outwards, and down again until it has become the outer rim.[16] Hermann von Hemholtz argued that, "As all our means of sense-perception extend only to space of three dimensions, and a fourth is not merely a modification of what we have but something

9 Manning 1956, 3.
10 Gardner, 2005, 151–152.
11 Möbius 1959.
12 Cayley 1959.
13 Cauchy 1959.
14 Richards 1988, 54.
15 Newcomb 1878.
16 Newcomb 1906.

perfectly new, we find ourselves by reason of our bodily organization quite unable to represent a fourth dimension"[17].

British mathematician Charles Howard Hinton played a key part in the popularization of 'hyperphilosophy' by publishing many writings during the years 1884–1907, speculating on the physical as well as spiritual aspects of 4-space. He also anticipated the hidden dimensions of string theory by stating that the fourth dimension could perhaps be observed on the smallest details of physical matter. Hinton coined the names *ana* and *kata*, which refer to the positive and negative directions along the axis of the fourth spatial dimension. Repurposed from the directional prefixes of classical Greek, these terms are widespread.[18]

Sometimes the concept of four-dimensionality was applied to three-dimensional space as a system of some other element than a point. The totality of lines[19], as well as spheres of varying radius[20] floating in a three-dimensional environment, can both be interpreted as a 'space' of four dimensions. After the success of Einstein's theories of relativity based on the four-dimensional concept of Minkowski space-time, the fourth dimension was widely regarded as time[21].

More recently, string theories in physics seeking to harmonize general relativity theory with quantum mechanics have aroused new interest toward higher-dimensional spaces. Although Finnish physicist Gunnar Nordström had already had the thought of a five-dimensional explanation for gravity, more renowned is the idea conjectured in 1919 by a Polish mathematician Theodor Kaluza, and developed by Swede Oskar Klein, that besides three 'extended' dimensions, there could be additional spatial dimensions 'curled up' into tiny cycles. The purpose of Kaluza-Klein theory was to provide a purely geometrical explanation of Einstein's relativistic gravity and

17 Helmholtz 1876, 319.

18 Other words used to refer to the directions sticking out to the fourth dimension have included e.g. "vinn" and "vout" – modifications of in and out, introduced by Rudy Rucker in his *Spaceland*. Unfortunately these terms are also quite misleading, as the fourth spatial dimension does not have anything to do with being in or out.

19 Plücker 1852.

20 Keyser 1911.

21 Consequently in the later occurrences the extra spatial dimension is sometimes called the fifth dimension, and bears occult connotations (see e.g. Bulgakov 1995, 214).

Maxwell's electromagnetism. The reason why the laws of nature appear to be different in different scales is because the small spatial dimensions have an effect on only the tiniest of subatomic particles. In string theories, the extra Kaluza-Klein dimensions themselves are interpreted as forming six-dimensional shapes called Calabi-Yau manifolds.[22]

Although the laws of nature arguably become simpler and more elegant when expressed in higher dimensions[23], it might be that only in three-dimensional space can there be creatures asking why there is a particular number of dimensions. The question is, of the possible universes of varying dimensions, which are more likely to contain observers?[24] One such possibility, equally as fictional as hyperspace, is a planar universe of just two spatial dimensions. The bearing it has on our discussion lies in its utility as a conceptual tool to understand the relation our space would have with respect to a four-dimension world.

FLATLANDS

The first step toward understanding the concept of hyperspace has usually come in the form of analogy. Perhaps surprisingly, it is instructive first to go one dimension down, and imagine a world with only two dimensions, inhabited by equally flat beings. Our challenges to understand and visualize four-dimensional space are analogous to the difficulties a two-dimensional being, confined to a plane, would have with respect to our three-dimensional space and its shapes. This idea can already be seen in Plato's allegory of the cave, where the prisoners experience only the two-dimensional shadows of a fuller, three-dimensional reality.

In fact, there is an entire tradition of 'flatlands', i.e. imaginary universes of only two dimensions, whose purpose is often to depict our own challenges with regard to non-Euclidean or higher-dimensional spaces. As James Joseph Sylvester wrote in his 1869 *A Plea for the Mathematician*: "... as we can conceive beings (like infinitely attenuated book-worms in an infinitely thin sheet of paper) which possess only the notion of space of the dimensions, so we may imagine beings capable of realizing space of four or a greater number of

22 Greene 2002, 184–209.
23 Kaku 1994, 8.
24 Pickover 1999, 230.

dimensions"[25]. An early representative of this genre is Gustav Fechner's 1846 essay *The Shadow Is Alive*[26], envisioning the life of a sentient shadow. In 1876 Hermann von Helmholz used the idea of two-dimensional beings to popularize the concept of the curvature of space[27], a method also demonstrated by William Kingdon Clifford's flat-fish[28].

The most famous of such fictitious conceptions is Edwin A. Abbott's 1884 classic *Flatland – A Romance of Many Dimensions*[29]. It depicts a satire of Victorian society through the life of polygons living in a plane. As already noted by Charles Howard Hinton (A Plane World, Scientific Romances, p. 129), Abbott's main focus was not on the geometry and 'conditions of life on the plane'. Hinton fixed this issue in his own writings *A Plane World* (1884) and *An Episode on Flatland: Or How a Plain Folk Discovered the Third Dimension* (1907). Hinton's planar world was called Astria, and it differed from Flatland with its lateral view that enabled a richer universe with heavenly bodies, gravity, etc.

Martin Gardner's introduction to Abbott and Hinton's two-dimensional ideas in his *Scientific American* article[30] was a herald for a new surge of interest toward the concept. Popular science authors like Dionys Burger[31], Jeff Weeks[32], and Ian Steward[33] used two-dimensional worlds and creatures to illustrate various topological and geometric qualities of surfaces. Mathematician and sci-fi author Rudy Rucker touched upon the problem of a two-dimensional world, also addressing some of the geometric errors and ambiguities in Flatland, e.g. in the visual perception of Abbott's polygonal protagonist[34].

Of the multiple successors of Flatland trying to grasp the peculiarities of fictional worlds restricted on a plane, the most ambitious is A. K. Dewdney. A computer scientist at the University of Western Ontario, his two-dimensional world, the Planiverse, first appeared in his 1979 article *Exploring the Planiverse*[35], and later in the same year in *Two-dimensional Science and Technology*, which he

25 Sylvester 1869, 238.
26 Translated for the *Mathematical Intelligencer* (Fellner 2011).
27 Helmholtz 1876, 303.
28 Clifford, 1886, 220.
29 Abbott 2006.
30 Gardner 1991b.
31 Burger 1965.
32 Weeks 2002.
33 Steward 2001.
34 Rucker 2002, 83–101..
35 Dewdney 1979.

published privately. Dewdney designed the Planiverse according to two principles concerning similarity and modification. Everything was required to be as much alike our three-dimensional world as possible, and in the situations where there were two conflicting hypotheses on different levels such as physics and chemistry, or chemistry and biology, the more fundamental one was prioritized over the more emergent one that is modified.[36] He developed his creation into a full novel in 1984 as *The Planiverse – Computer Contact with a Two-Dimensional World* [37], an account of a planar universe complete with its physics, chemistry, biology, politics and art.

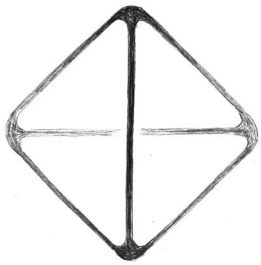
Careful observation of the conditions of a life in the plane is a most instructive approach in the early introduction to the concept of higher space by analogy. A reader understanding the difficulties of flatlanders in grasping our three-dimensional shapes, and empathic toward them in this regard, is well prepared to encounter the *polychora* – a set of hyperspatial structures portrayed in the present work.

THE POLYCHORA

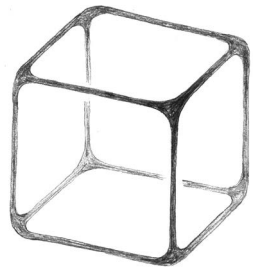
For a given space, it is natural to ask what kind of geometric shapes can exist there? Special attention has been given to the *polytopes*, whose rigorous definition, however, falls outside the scope of the present study. For us it suffices to say that they are geometric shapes built from straight, or 'flat' parts of varying dimensionality. Their zero-dimensional parts are called *vertices*, their one-dimensional parts are the *edges* that are bounded by the vertices, the two-dimensional parts of a polytope are called *faces*, and they are bounded by the edges. Taken together, these parts comprise the *elements* of a polytope.[38]

36 Gardner 1997, 4.
37 Dewdney 2000.
38 Furthermore, sometimes it is convenient to discuss the *facets* of a polytope, which are the $n-1$ elements of an n-dimensional polytope. The facets of a polygon are its edges, facets of a polyhedron are its faces, and so on. If the quantities, shapes and connections of the facets are known, the details of all the other elements can be deduced from that information. The number of the facets gives the polytopes their Greek names.

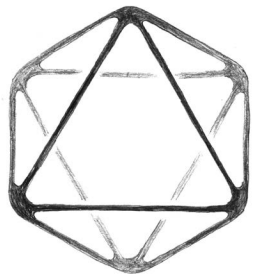
The tetrahedron



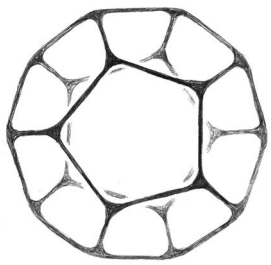
The cube



The octahedron



The dodecahedron



The icosahedron

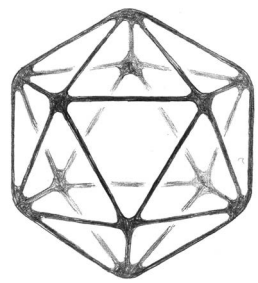


FIGURE 0.1: The Platonic solids

To narrow the scope further, we can consider only *regular* polytopes, in which the elements having the same dimensionality are identical. Thus in two-dimensional space all the regular polygons are considered regular polytopes, whereas in three-dimensional space the polytopes fulfilling the criterion comprise the renowned Platonic solids (FIGURE 0.1).

Given the requirement for the regularity that all the elements of the same dimensionality should be identical, there exist exactly five polyhedra: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron.

The *tetrahedron* is composed of four vertices, six edges, and four triangular faces. There are three faces meeting at each vertex.

The *cube* is composed of eight vertices, twelve edges, and six square faces. There are three faces meeting at each vertex.

The *octahedron* is composed of six vertices, twelve edges, and eight triangular faces. There are four faces meeting at each vertex.

The *dodecahedron* is composed of twenty vertices, thirty edges, and twelve pentagonal faces. There are three faces meeting at each vertex.

The *icosahedron* is composed of twelve vertices, thirty edges, and twenty triangular faces. There are five faces meeting at each vertex.

Between these polyhedra there exists a significant relation called a *duality* – a correspondence of the faces and vertices observable already in the numeric information above. Thus the octahedron and the cube are duals of each other, the dodecahedron and icosahedron form another dual pair, whereas the tetrahedron is observed to be *self-dual*.

As noted above, points, lines, planes, and volumes (3-spaces) all bound each other in a similar manner in the hierarchy. From a line we can secure a bounded 'region' with two zero-dimensional points, getting a line segment. Line segments can be used to secure a bounded region from a plane: a polygon. Polygons, again, can be used to bound a three-dimensional region from 3-space, a polyhedron. This hierarchical logic already offers a seed for four-dimensional thought

in the form of a natural question: can polyhedra somehow be joined together to immure a bounded region from a higher space yet?

To see how and why the answer is in the affirmative, it is instructive to inspect the situation one dimension lower and notice that it is the angle at the edges that affords the three-dimensionality of the polyhedron. Similarly, if two polyhedra are sitting in a four-dimensional space, they can be joined at their faces *at an angle*. Now each polyhedron lies in its own three-dimensional slice of the 4-space, and they meet at the ridge-like face.

Just as with the polygonal faces on the Platonic solids, the right kind of assembling of the polyhedra will eventually close in on itself and form a four-dimensional polytope. These hypersolids are called polychora, and the polyhedra acting as their facets are called their *cells*. In his 1852 *Theorie der vielfachen Kontinuität* Ludwig Schläfli proved the existence of the 'polyschemes', six regular polychora analogous to the Platonic solids in 3-space, although the work was not published until 1901. Let us next take a closer look at these highly symmetric four-dimensional structures and some standard methods of constructing them.

The following illustrations are schematic, graph-like diagrams, where the lengths and angles are chosen to produce a symmetric composition laid out on the paper. However, the pictures below favor the planar clarity of the representation at the expense of geometric accuracy; they demonstrate many of the original features of the structures. With some concentration it is even possible to find all the three-dimensional facets of the polychora in the tangle of lines that comprise the final stage of each drawing.

FIGURE 0.2 shows how to build higher-dimensional polytopes by adding new vertices. Starting from a point (a), we can add another point (b) outside the zero-dimensional space of the original point. Joined together they make a line segment (c). We can add another point again (d), now outside the one-dimensional space of the line segment. This point joined to the line segment makes a triangle (e). Adding a new point (f) outside the two-dimensional space of the triangle makes a tetrahedron (g), and finally a point (h) outside the three-dimensional space of the tetrahedron takes as to the first four-dimensional polytope, the *pentachoron*, also called the 5-cell (i). It has five vertices, ten edges, ten triangular faces, and five tetrahedral cells. There are three cells meeting at each edge, and four at each

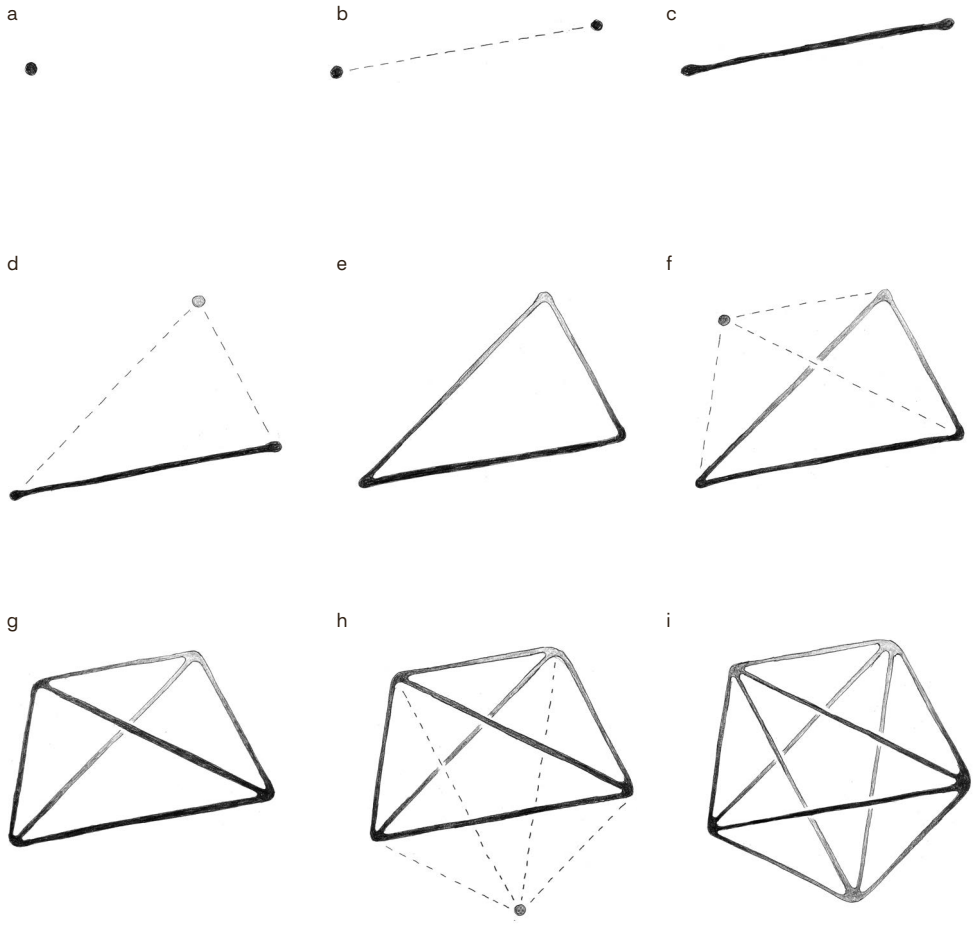


FIGURE 0.2: Constructing the pentachoron

vertex. As this construction can be continued indefinitely onwards, the resulting family of polytopes extends to all higher dimensions. These structures are collectively called the *simplices*.

In FIGURE 0.3 we see another way to arrive at a four-dimensional solid by dragging the lower-dimensional shape perpendicularly away from its space. If a point (a) is moved (b), it sweeps a line segment (c). If that line segment is then moved perpendicularly away from its line (d), over the distance of its length, it traces out a square (e). Similarly, we get the cube from the square (f,g), and finally when the cube is moved perpendicularly away from its 3-space over the distance of its edge length (h), the trace is a *tesseract*, also called the 8-cell (i). It has sixteen vertices, thirty-two edges, twenty-four square faces, and eight cubical cells. There are three cells meeting at each edge, and four at each vertex. This family of polytopes also extends to all higher dimensions, and they are collectively called the *hypercubes*.

Yet another method is to place points on mutually perpendicular axes (FIGURE 0.4). Starting from two points on a single axis, we can erect another axis perpendicular to the first (a). Equidistant points on it give us a square (b). A third perpendicular axis with two points on it (c) yields an octahedron (d). Finally, a fourth axis perpendicular to all the previous ones is erected, and the two points on it connected to the vertices of the octahedron (e) gives us the four-dimensional solid called the *hexadecachoron*, or the 16-cell (f). It is composed of eight vertices, twenty-four edges, thirty-two triangular faces, and sixteen tetrahedral cells. There are four cells meeting at each edge, and eight at each vertex. Again, this family of polytopes also extends to all higher dimensions, and they are collectively called the *orthoplexes*.

Our discussion here will also feature a more intricate regular solid in four-dimensional space: the *icositetrachoron*, also called the 24-cell. It is composed of twenty-four vertices, ninety-six edges, ninety-six triangular faces, and twenty-four octahedral cells. There are three cells meeting at each edge, and six at each vertex. It is uniquely a four-dimensional conception, as it does not have any regular analogs in lower or higher dimensions. There are two even more intricate structures in four-dimensional space – the *hecatonicosachoron* (120-cell) and *hexacosichoron* (600-cell). The hecatonicosachoron is composed of 600 vertices, 1200 edges, 720 pentagonal faces, and 120 dodecahedral cells. It has three cells meeting at each edge, and four at each vertex.

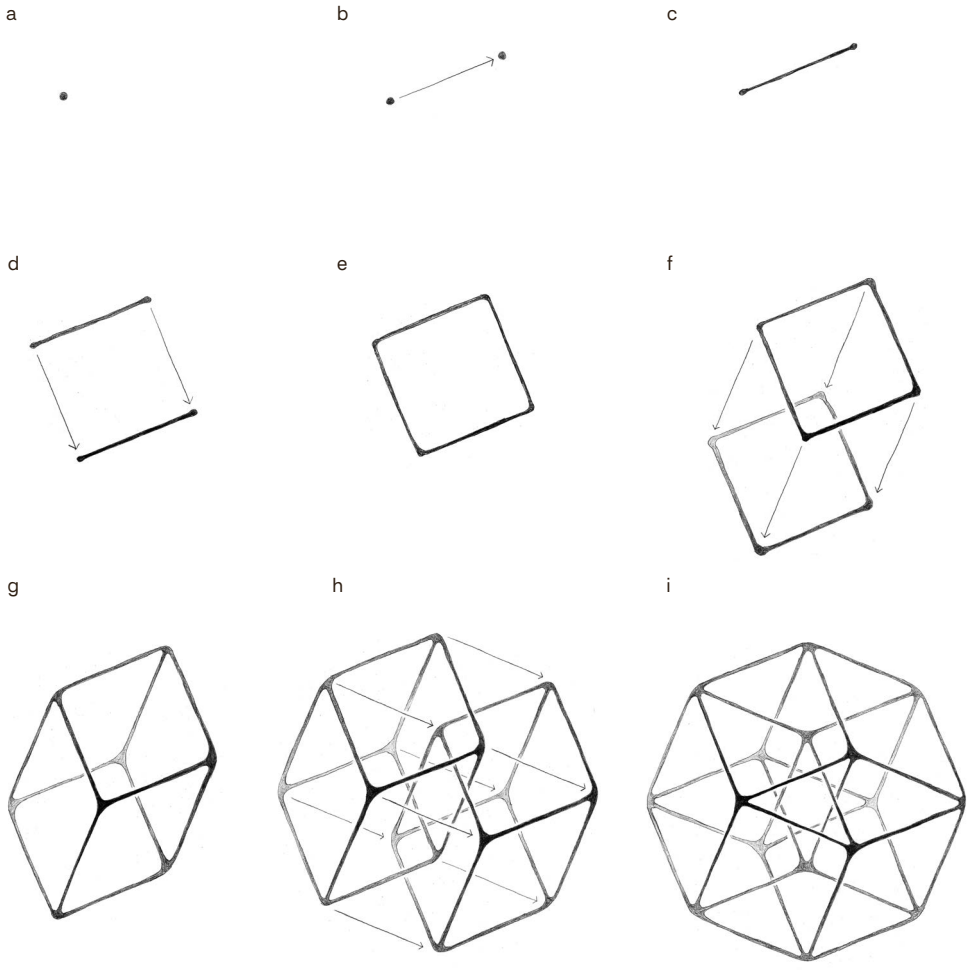


FIGURE 0.3: Constructing the tesseract

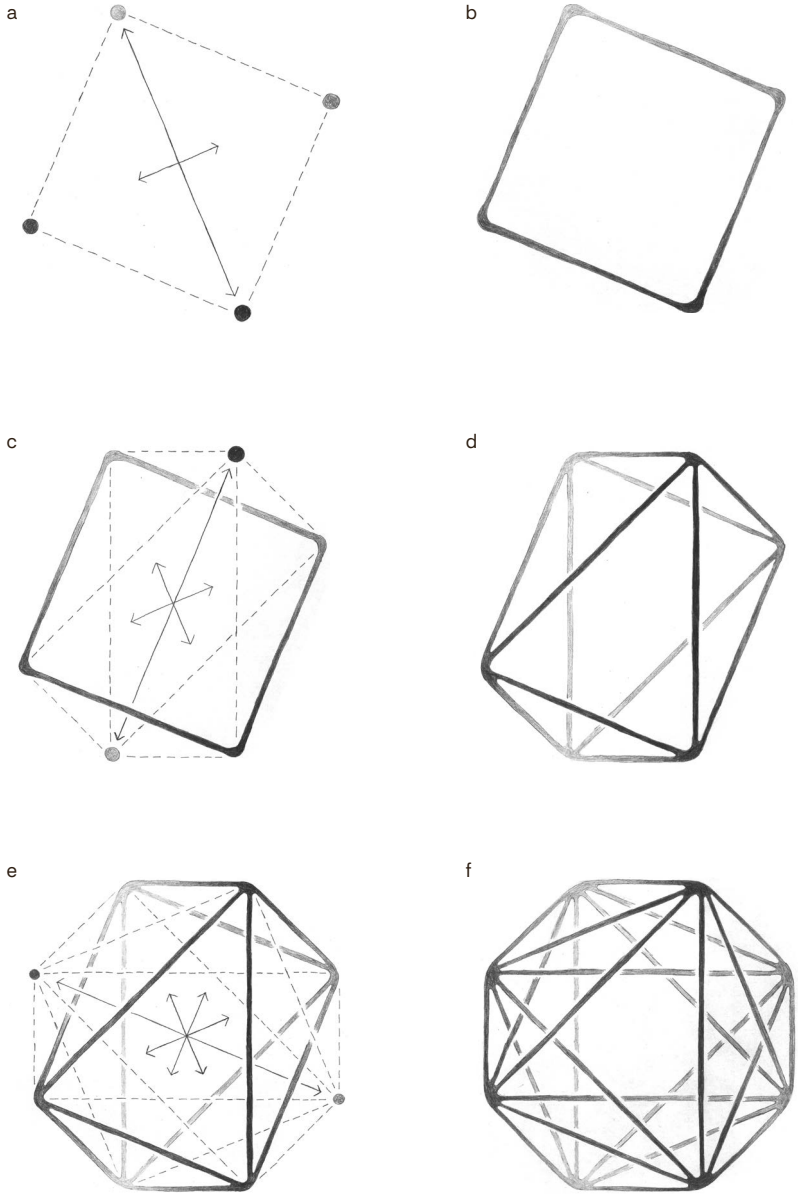


FIGURE 0.4: Constructing the hexadecachoron

The hexacosichoron is composed of 120 vertices, 720 edges, 1200 triangular faces, and 600 tetrahedral cells. There are five cells meeting at each edge, and twenty at each vertex. As seen in the quantities of the elements, the hecatonicosachoron and the hexacosichoron are duals of each other. They can be seen as analogs of the dodecahedron and the icosahedron, but they do not have analogs in higher dimensions.

As noted already by Schläfli, in each *n*-dimensional space where the number of dimensions (*n*) is five or higher, there exist only three regular structures: the *n*-simplex, the *n*-hypercube, and the *n*-orthoplex. If the focus is shifted slightly to discuss the symmetry groups afforded by n-dimensional spaces instead of structures of Platonic regularity, mathematics can describe phenomena such as the "Monster Group", existing exclusively in a space of 196,883 dimensions.[39] I am excluding dimensions higher than four from the present work, as the flattening required to produce visual representations of them becomes too severe for comprehensive visual reference.

During the investigation we will also touch upon some of the *semiregular* polytopes, which are allowed to have different kinds of facets. An English lawyer, Thorold Gosset, classified them as a hobby in 1897[40], and they were rediscovered in 1911 by E. L. Elte[41].

VISUAL INTERPRETATIONS

Because four-dimensional structures cannot be directly portrayed in our physical world, the focus of many inquiries into the subject has been on the challenge of developing a visual understanding of four-dimensional space. The mathematical visualizations have usually been produced for educational purposes, but the concept has also roused interpretations in fine arts. As these categories are often difficult to define and are perhaps even artificially separated, I have chosen to give the references below in a chronological order instead of a topical one.

Working in the fields of geometry and topology, there have been many mathematicians with a special penchant for drawing careful images of their subject matter, and the work they put into

39 Parker 2014, 344.
40 Coxeter 1973, 164.
41 Coxeter 1973, 210.

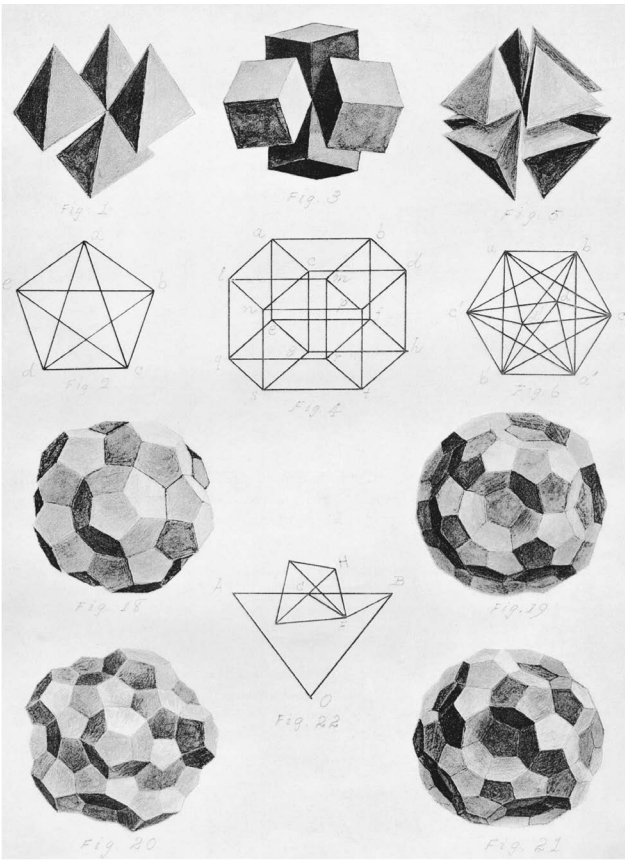


FIGURE 0.5: Stringham's illustrations of the polychora

visualizing four-dimensional structures is particularly instructive[42]. The first pictures of four-dimensional solids were created by Irving Stringham in his 1880 dissertation under the advisement of James Joseph Sylvester at John Hopkins University. He gave an account of the six regular polychora and produced beautiful illustrations of their unfoldings, where the cells of the polychoron are split apart and rotated until they lie on the same three-dimensional slice.[43]

The first perspective pictures of the regular polychora appear in Victor Schlegel's 1882 paper for the Société Mathématique

42 For a detailed account of the early visualizations of four-dimensional geometry by Stringham, Jouffret, Schlegel, Hall, Schoute, mentioned only briefly below, I recommend my reader to consult Tony Robbin's 2006 work *Shadows of Reality*.

43 Stringham 1880.

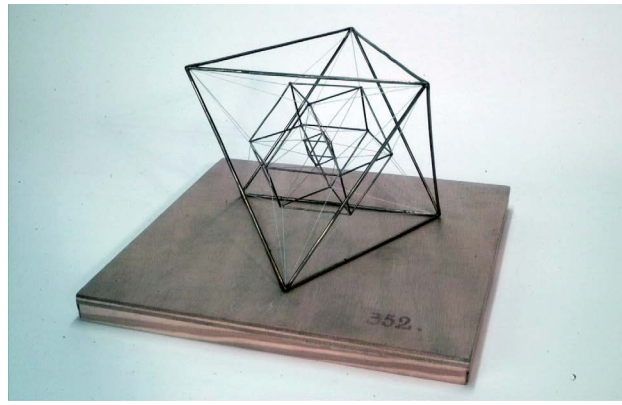


FIGURE 0.6: Schlegel's model of the icositetrachoron at the University of Göttingen

de France. Soon after he began to work on building three-dimensional models of the same subject matter out of steel wire and silk thread. These objects were designed as perspective projections where the projection point lay just beyond the center of one of the facets. Projected like this, the cells of the polychora are arranged neatly in concentrically nested layers, and none of the faces crash into each other.[44] Schlegel's models were sold in Ludwig Brill's[45] and Schilling's[46] catalogs. Today, the University of Göttingen[47] has a collection of these models on display.[48]

Charles Howard Hinton developed a mnemonic system of some tens of thousand cubes with individual names in Latin, serving as a three-dimensional mental retina of a kind on which to visualize the successive cross-sections of objects in 4-space[49]. Interested in Eastern thought, he also sought to eliminate the 'self elements' of his system by memorizing the different orientations and mirror reflections of the cubes. Later he developed the system into a self-help method to visualize the fourth dimension, which consisted of manipulation of colored cubes. The cubes were available for purchase from his publisher.

44 Such diagrams of polytopes later became known as Schlegel diagrams.
45 Dyck 1892, 253–254.
46 Schilling 1911, 156.
47 Göttingen collection of mathematical models and instruments (modellsammlung.uni-goettingen.de).
48 Robbin 2006, 11–13.
49 Hinton 1888.

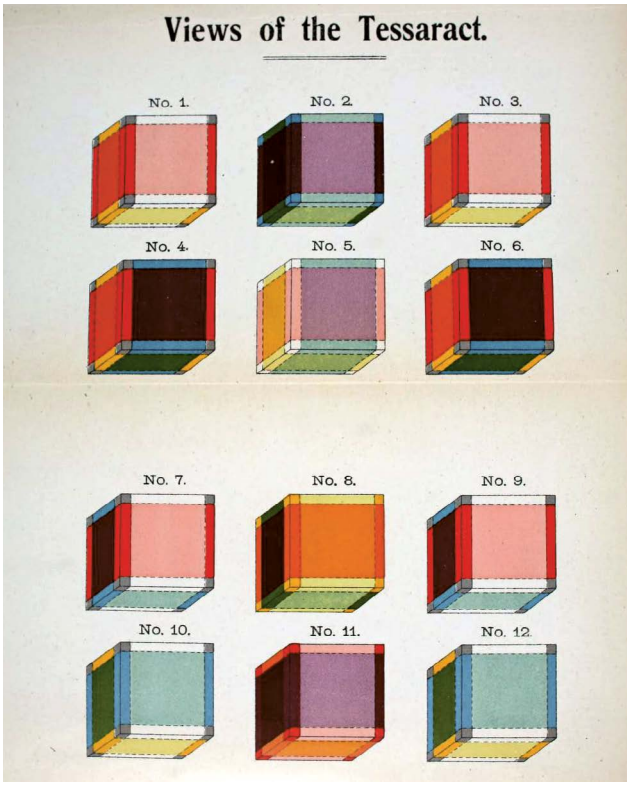
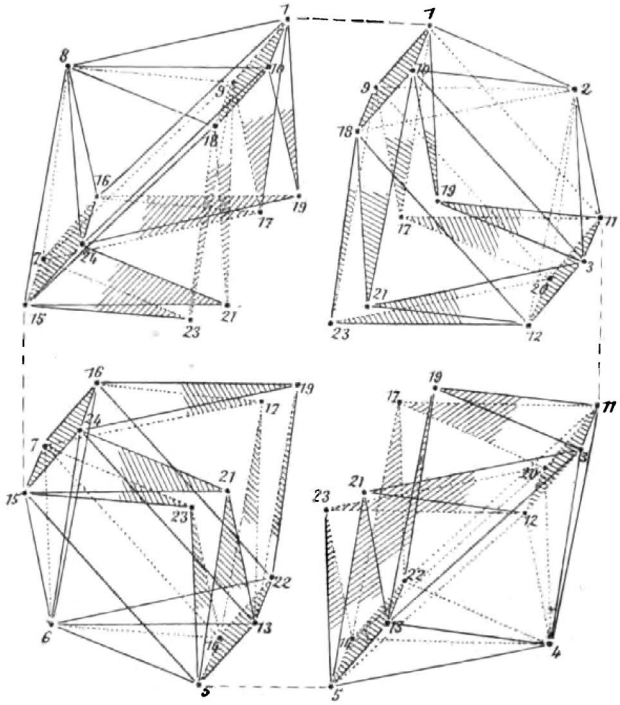


FIGURE 0.7: The frontispiece of Hinton's *The Fourth Dimension* (1901)

Hinton was a frequent guest at the household of Mary Everest Boole, whose husband George was famous for his Boolean algebra. During these visits he used Alicia, Boole's young daughter, as the primary guinea pig for his system of cubes, an activity encouraged by her mother who was also known for her writings on early mathematics education. Alicia showed a special talent for visualizing the fourth dimension, a skill in which she soon exceeded that of Hinton himself. Despite her restricted circumstances as a housewife without any sort of formal mathematical training, Alicia Boole Stott went on to independently prove the existence of the six regular polychora, describe their perpendicular cross-sections, and also find some of the semiregular polytopes in four dimensions. She also discovered some of the cross-sections of the hypersolids that Schoute was working on, and later the two worked together in collaborations where Boole Stott's rare talent for visualization was paired with Schoute's analytical

FIGURE 0.8: Jouffret's exploded view of the icositetrachoron



methods. From the 1930s she also collaborated with Coxeter, studying four-dimensional polytopes[50] and their cross-sections. During her life she also made many colored cardboard models of the polychoral cross-sections. Today some of these interesting objects are held in the museum of the University of Groningen.[51]

Building upon Gaspard Monge's invention of *Geometria Descriptiva* a decade earlier, Dutch mathematician Pieter Hendrik Schoute produced mechanical drawings of the four-dimensional polytopes for his 1902 *Mehrdimensionale geometrie*[52]. These methods were later elaborated by in France by Esprit Jouffret, whose drawings depicted the polychora in exploded views and projected down to three dimensions[53]. In his *Shadows of Reality* Tony Robbin gives a speculative but vividly convincing storyline of how Jouffret's see-through

50 The word "polytope" is Boole Stott's coinage.
51 Blanco 2014.
52 Robbin 2006, 9.
53 Jouffret 1903.

illustrations of the hypersolids found their way into the hands of Pablo Picasso, and became a major influence of his early cubistic works[54].

The discussion of the influence of the fourth dimension on fine art belongs to the domain of art history, and consequently falls outside the scope of this study. A reader interested in these aspects should consult the works of Linda Dalrymple Henderson, especially *Fourth Dimension and Non-Euclidean Geometry in Modern Art* (1983). It suffices to note that for modern art, the 4-space served as an inspirational basis and argument in the agendas aimed at reaching higher visual reality free from earlier conventions like linear perspective. In particular, Hinton's ideas were echoed in the writings of those such as theosophist P. D. Ouspensky[55] in Russia and Henri Poincaré[56] in France, and thus contributed to the development of modern art movements such as cubism. Particularly explicit is the hyperspatial inspiration for Marcel Duchamp's seminal 1915–1923 work *The Bride Stripped Bare by Her Bachelors, Even* (*The Large Glass*), as described in his own notes on the process[57].

In the United States architect Claude Bragdon contributed to the subject with designs as well as written works. Having met Hinton through their mutual friend Gelett Burgess,[58] Bragdon's carefully illustrated *A Primer of Higher Space*[59] and *Explorations into the Fourth Dimension*[60], originally published as *Four-Dimensional Vistas* (1916), served to further popularize the concept. His work went on to inspire Buckminster Fuller's involvement with four-dimensional space in the sixties, although his interpretation of the concept in the context of his own theorization of kinetic systems was somewhat idiosyncratic[61]. Prompted by a series of provoking dreams, an American rug seller, Paul S. Donchian, started to build wire models of the four-dimensional polytopes in order to teach himself four-dimensional geometry. These amazingly intricate models, which even included the hecatonicosachoron (FIGURE 0.9), were shown in 1934 at the *Century of Progress Exposition* in Chicago, and at the *Annual Exhibit of the American*

54 Robbin 2006, 28–40.
55 Henderson 2013, 4–5.
56 Hendersson 2013, 25.
57 Hendersson 2013, 250.
58 White 2018, 108.
59 Bragdon, 1913.
60 Bragdon 1972.
61 Henderson 2013, 42–43.

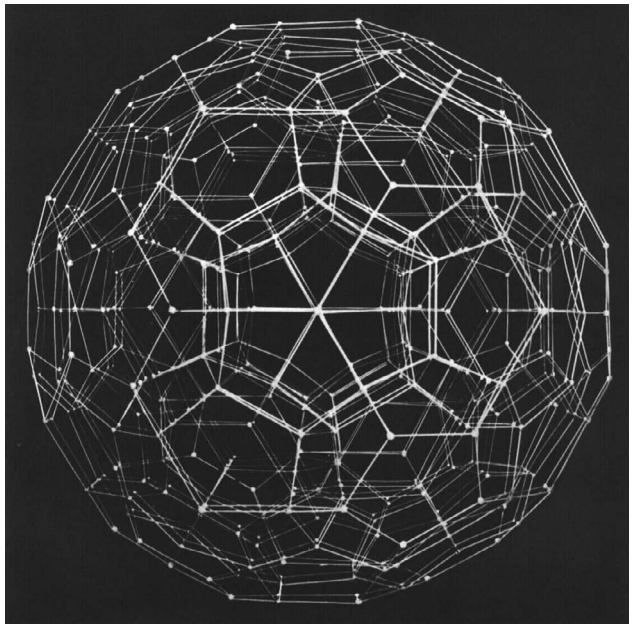


FIGURE 0.9: Parallel projection of the hecatonic-osachoron by Paul S. Donchian

Association for the Advancement of Science, in Pittsburg.[62] Canadian polymath T. P. Hall's mention of a tesseract model he built from hinged vertices and telescopic edges has a particular bearing on the present study, as it was capable of portraying a rotation about a plane[63].

More recently the availability of digital visualization technologies has given artists a chance to study higher space with a greater fidelity to the precise geometry of the concept, as Michael A. Noll's computer animation[64] of a rotating tesseract in 1966 marked the beginning of a new era in four-dimensional visualization. A decade later at Brown University, Charles Strauss and Thomas Banchoff produced *The Hypercube: Projections and Slicing*, an animation of a tesseract spinning in 4-space, which was shown at the International Congress of Mathematicians in Helsinki[65].

One precedent of the objects pursued in the present work is Banchoff's paper model of the unfolded tesseract, which allows some of the faces to be folded back against each other. Having

62 Coxeter 1973, 260–262.
63 Robbin 2006, 15.
64 Noll 1978.
65 Banchoff 1996, 67.

featured an unfolded tesseract in his 1954 painting *The Crucifixion* (subtitled *Corpus Hypercubicus*), this model was of great delight to Salvador Dalí, who received it as a gift from Banchoff. The model was reworked to a large scale motorized copy for the Dalí museum in Figueras, Catalonia.[66]

A groundbreaking work in tactile four-dimensional visualization in the U.S. was that of David Brisson, an artist on the faculty of Rhode Island School of Design, who began producing projections of the regular polychora as sculptural wire models in 1960[67]. He also produced stereoscopic pictures[68] of four-dimensional shapes, in which he experimented on using the offset for acquiring an illusion of a hyperdepth. In the same vein he developed a four-dimensional visualization technique of 'hyperanaglyps'[69] that consisted of a red and blue pair of sculptures superimposed together and placed on slowly moving turntables to observe stereoscopic glasses. His partner Harriet Brisson built sculptures of plexiglass and two-way mirrors that were inspired by the pentachoron and the hexacosichoron[70]. Together the Brissons organized symposia and exhibitions under the title of *Hypergraphics* in the late seventies.

Perhaps the most extensive contribution on the subject was made by Tony Robbin, who has also written extensively on the subject. His paintings had welded rods sticking out of them, creating a hyperspatial effect when mixing with the polychoral ornaments painted in acrylic on the canvas. Like David Brisson, he also used stereoscopic effects achieved through installations of colored lights and shadows.[71] Informed by Scott Carter's mathematics of braiding surfaces in 4-space[72], Robbin has also constructed multilayered paintings with interweaved patterns[73].

Shorter mentions can be given to painter Toshi Katayama, who had two projections of the tesseract in octagonal symmetry rotating around a pin in the center to portray different arrangements of the cubical cells[74]. Attilio Pierelli, an Italian sculptor of the *Dimensionalismo*

66 Banchoff 1996, 106.
67 H. E. Brisson 1993, 41.
68 D. W. Brisson 1978.
69 H. E. Brisson 1993, 40.
70 H. E. Brisson 1993, 43–44.
71 Robbin 2011, 52.
72 Carter 2004.
73 Robbin 2015.
74 Miyazaki 1986, 93.

movement, made a tesseract central perspective projection from stainless steel in 1974[75], and a parallel projection of the tesseract that can be found in the Kulosaari district of Helsinki, where Håkan Simberg's steel sculpture *Tesseraktin varjo* was inaugurated in 2000.

New digital technologies have made it possible to develop various interactive applications whose pedagogical value lies in the possibility of manipulating the visualized shapes in four dimensions. George Francis, a University of Illinois mathematician famous for his special knack for visualization – with blackboard as well as with computers – has researched and developed *RTICA*s (an acronym for *Real-Time Interactive Computer Animation*) Many of the homotopies (smooth changes-of-shape happening over time) depicted by the applications, often built in collaboration with his students, have depicted four-dimensional phenomena, such as turning a sphere inside out[76].

A freelance mathematician and MacArthur prize recipient, Jeff Weeks, has worked on the development of various interactive software for visualizing topology, non-Euclidean geometry, and four-dimensional space. His *4D Draw*[77] application features a cubical scene where the user can construct line segments bounded by points. Besides the ordinary three coordinate values the points have also a color, which reveals their position along the fourth dimension. The idea of using color to visualize the fourth dimension had already been proposed in 1928 by Hans Reichenbach[78]. Another piece of software, *Curved Spaces*[79], is relevant to our discussion here as it offers a flight simulation through the three-dimensional surface of the hypersphere – among other multi-connected universes of varying topology.

Marc Ten Bosch has developed an app, *4D Toys*, released in 2017 for mobile devices, which allows the interactive exploration of the cross-sections of various four-dimensional objects. Currently in development is Bosch's full game *Miegakure*[80], where the player's view can be rotated in 4-space to gain access to a parallel three-dimensional slice and to afford a passage to a region confined in the previous three-dimensional slice. A curiosity perhaps, but worth a mention is

75 Banchoff and Cervone 1993, 91.
76 Francis 2005.
77 Weeks 2014.
78 Reichenbach 1958, 280–283.
79 Weeks 2009.
80 Bosch 2010.

the four-dimensional version of Rubik's Cube[81], which Melinda Green has also implemented as a physical, hand-held puzzle[82]. The studies of human perception and spatial navigation in virtual environments of four or more dimensions have yielded promising results in the fields of psychology and cognitive science[83].'

The developments in computing applications have led back to sculptural form in four-dimensional visualization through the employment of digital design techniques. It is remarkable that the geometry of the *Zometool* construction set affords a construction of the parallel projections of the polychora. In particular, George Hart, a research professor in the engineering school at Stony Brook University, has worked on student projects constructing intricate models of various four-dimensional structures on a large scale[84], as well as during his visit to Aalto University. Scott Vorthmann's software application *vZome*[85] completes the set with the vertex angles missing in the physical *Zometool*. Robert Webb's software *Stella4D*[86] has a library of many uniform shapes and allows the modification of the shapes by e.g. truncation. The application lets its user export paper model template files ready for cutting, folding, and gluing. With the help of his software, Webb has built an attractive paper model[87] that shows a toroidal decomposition[88] of the hecatonicosachoron.

Finally, serious efforts in employing the digital technology of 3D printing into visualization of four-dimensional geometry have been undertaken by Carlo Séquin[89] and in particular Henry Segerman[90], whose models can be acquired through *Shapeways*, an online 3D printing service[91]. Working with Saul Schleimer for the *Brilliant Geometry* exhibition at the University of Warwick, Segerman also designed a zoetrope consisting of three-dimensional printed frames of the tesseract undergoing a double rotation, portrayed in stereographic projection.

81 Yoshino 2017.
82 Green 2016.
83 See e.g. Aflalo and Graziano 2008.
84 Hart 2007.
85 Vorthmann 2017.
86 Webb 2007.
87 Webb 2016.
88 Banchoff 2013.
89 Séquin 2002.
90 Segerman 2016.
91 (www.shapeways.com/designer/henryseg).

We have now seen how the previous visual interpretations have fallen under the categories of planar visualizations, consisting of drawings and paintings, virtual models such as animations and software, and physical objects such as sculptures, toys, and various geometric models. Two-dimensional illustrations of four-dimensional structures have the downside of having gone through not one, but *two* operations of flattening. On the other hand, although computer-generated animations have the advantage of showing the deformations of the projection caused by the movement of the projected shape in 4-space, as a two-dimensional medium they lack the tactile and plastic availability of three-dimensional objects. Three-dimensional objects are arguably valuable tools in getting a visual understanding of hyperspatial geometry, especially if they possess some functional effect that further illuminates the phenomena.

OBJECTIVES

Positioned between mathematical visualization and art education, the objective of the research reported here is to craft physical, three-dimensional models that illustrate four-dimensional structures, namely the regular polychora. Through artistic research, the purpose is to design pedagogical tools with which to demonstrate various hyperspatial phenomena. I especially considered kinetic models based on the four-dimensional regular polychora to be beneficial for understanding these structures, although hyperspatial operations, such as the rotation around a plane, are difficult to implement because of the distortions resulting from the required flattening.

My focus was narrowed naturally in finding novel designs. I was interested to know what kind of actions the concept of four-dimensional space afford us in the context of visual practice, and whether a scholarly study of the geometry and topology of dimensional space could enrich the morphological repertory of visual art practice. As I have earned my MA degree in art education, I approached my goals mainly from this perspective. In the context of the present study it means remaining confidently on the *visual* side of the story. The visuality of the phenomena is consequently taken seriously and considered as a valid research subject in its own right. Consequently the occasional research questions take the form of "If I would do this or that, what would the result *look* like?" I was also curious to know what kind of novel modes of spatial reasoning might arise from four dimensionally informed visual practice. Through what kinds of visual effects are the four-dimensional interpretations and insights evoked?

As pedagogical tools, the models pursued offer a hands-on experience of hyperspatial geometry. The pedagogical standpoint also prompted me to prefer construction experiments of models crafted with classic, or traditional techniques, instead of technology-driven projects. Although I used 3D modeling software to help design the artifacts, the end results are made of simple techniques and materials. This approach set me aside from the field of mathematical visualization, which relies heavily on digital technologies and is eager to adopt newly found technical inventions such as 3D printing or virtual reality.

The motivation for the research presented here arose from working with visual aspects of geometry and topology both as an art student and as a teacher. When I previously explored the geometric and topologic possibilities of knot ornaments and designed a constructive

interlace ornamentation concept in two- and three-dimensional environments[92], I hit upon the limitations of our three-dimensional reality and was thus introduced to the concept of the fourth dimension of space. This precursory work gave me confidence in the potency of artistic research to investigate interdimensional questions pertaining to geometry and topology. The planning and lecturing of interdisciplinary courses in Aalto University in collaboration with mathematicians and architects further developed my curiosity toward these problems, and my lecturing on these subjects has enabled me to explore them from a pedagogical perspective. The desire to develop pedagogic methods to apprehend space of higher dimensions arises from these experiences.

Digital visualization technologies such as computer assisted modeling, game design, 3D printing, algorithmic design, and virtual reality have made it possible to make visual productions without previous education in the arts. Indeed, computer-generated two- and three-dimensional visualizations like animations and 3D-printed objects are excellent means of swiftly getting an idea of the visuality of the phenomenon at hand. Often, however, they lack the completeness associated with works of art that arises from careful consideration and combination of formal elements of the piece, like material, technique and style. The visual argumentation in publications within the interdisciplinary field of art and mathematics often consists of illustrations exported straight from the mathematical software, and the applications used to produce the mathematical content are not necessarily equipped with sufficient features for high quality visual output. It must also be recognized that computer-generated imagery carries stylistic bearings and contextual associations as strong as any other graphic or plastic medium.

With my background in visual arts, I also hope on my part to improve the overall visual quality of hyperspatial imagery. Even if a full understanding of the artifacts requires some mathematical knowledge of higher-dimensional space, I hope that the models will also evoke immediate visual attraction, even for the lay audience. Nevertheless, my objective was to design objects that make a real contribution to comprehension of higher space, as opposed to complex 'eye candy' capable only of confusing and frustrating a spectator eager to understand the actual phenomena.

92 Luotoniemi 2011.

METHODOLOGY

The research reported here is an *artistic research*. This relatively new family of disciplines differs from art history, art theory, aesthetics, and philosophy of art, in that art is not the subject of inquiry, but rather a method. A good definition of artistic research is that instead of being research *of* art, it uses artistic practice in researching something else. Here the expression 'artistic' is understood foremost in a very naïve manner, roughly translating as 'visual'. I describe my practice as being artistic, since at every turn I was mainly concerned about what things look like when navigating through the phenomena. Another artistic feature of my work also pertains to the inclusion of the poetic aspects of the mathematics into the research process.

EXPERIMENTAL CRAFTING

I am investigating the questions related to the visuality of the fourth spatial dimension through experimental artistic research incorporating methods of visual arts, like drawing and wire model making, and facilitated by 3D modeling computer applications. These constructive experiments are based on the geometries of regular four-dimensional polytopes.

Of those Finnish philosophers writing on artistic research, Tuomas Nevanlinna, for example, has recognized the experimental nature of artistic research, distinguishing it from research in humanities[93]. With his twenty years' involvement in the development of artistic research methodology from its early conception, Juha Varto also stresses the experimental aspect of the practice as a strategy of making the phenomena appear in an unexpected manner that can resist anticipated contextualizations[94].

To develop and design the models that are the objective of this research, I employed the simple principal of dimensional analogy. Luckily, just as three-dimensional structures can be drawn, unfolded, sliced, photographed or otherwise projected onto a two-dimensional medium like paper or computer screen, these graphical techniques can be generalized to get the three-dimensional appearances of polychora described above. The theory guiding these constructions comes from the field of descriptive geometry, which has traditionally considered the projections of a three-dimensional object on a plane.

93 Nevanlinna 2002, 67–71.
94 Varto 2018, 73.

By dimensional analogy, the same principles hold when representing any object of n dimensions in an environment of n-1 dimensions as well. Often, before moving on to the next rung of the ladder, we will stop to examine various appearances of the lower-dimensional polytopes to facilitate the interpretation of the projections of the higher-dimensional ones. Although the observations made from the lower-dimensional situations might seem self-evident, they will show their utility when reflected against the higher-dimensional settings looming ahead. I will often even repeat the actual wordings of the arguments to emphasize this correspondence. An efficient example of this approach is Scott Kim's development of a four-dimensional version of the famous illusion of the Penrose tribar[95]. Unlike many of the current visualizations, the projections depicted by my models are not constructed numerically from four-dimensional Cartesian coordinates, but are built synthetically. We will see how the symmetries of the lower-dimensional elements will usually determine the shape of the higher-dimensional structures as seen at a particular viewpoint.

In contrast to the vast majority of contemporary contributions in four-dimensional visualization, the investigation reported here is not technology-driven. I am excluding all technologically advanced, mechanical or digital implementations of my research projects. Firstly this position gives me a research niche that is currently underexplored. I want to use simple, technologically minimal designs that are currently lacking from the field. As a host of e.g. four-dimensional computer visualizations already exist in the field, it would not be appropriate for me to pursue them here. Considering my background, such productions would undoubtedly suffer by comparison with the existing ones. Secondly, my technologically restrained position also arises from pedagogical considerations. I think that elaborate mechanical or digital technology would distract my audience from the concepts at hand, and mislead them into thinking that the topics themselves are somehow dependent on technology. In the end it is an intriguing challenge: can simple models made of wire, sticks, paper and cloth alone induce the otherworldly experience of hyperspace?

Considering the above, and to distinguish my models from those produced in plain mathematical visualization, I refer to

95 Kim 1978.

them as being *crafted*. However, I do not wish to imply a craft like that of an artisan, in which there is an evident virtuosity resulting from years of practice. We will see how my models actually bear a closer resemblance to the 'arts and crafts' of a hobbyist. The somewhat unpolished appearance of the objects is intentional, as it serves to emphasize the playfully experimental nature of the project.

In this report, however, I have deliberately disregarded those aspects of my process that I believed to pertain to crafting in general. This is not because I consider them mundane, primitive, or not worthy of the name 'research'. Instead, as it was the first time I had ever glued together a paper model, operated a sewing machine, or tied some sticks to a configuration, it is unlikely that I could have contributed anything on these matters beyond the most elementary insights. The struggles I had with these crafting techniques were arguably characteristic to them in general, and the precisely four-dimensional point of departure of the projects discussed here would not add any special value to scholarly investigations on them.

INCLUSION OF THE POETIC

As Varto notes, it is precisely the experimental method that takes the fictitious nature of a research project seriously[96]. In the context of artistic research, fiction is a structure, which although it could be, is not necessarily real[97]. Although some of these constructions are more convincing than others, fiction, taken seriously, can act as a point of departure for a research project[98].

In the investigation at hand, there is a two-fold fictionality manifesting itself. What makes every enquiry into four-dimensional geometry a fictitious endeavor is the fact that the whole notion was originally conceived as a thought experiment not required to have any application to reality. Although such applications exist, it is not a prerequisite for investigations into the subject. This seemingly self-contained characteristic present in all pure mathematics has given mathematicians a reason to think of their work as being research of a world with an independent existence[99].

96 Varto 2018, 74.
97 Varto 2018, 70.
98 Varto 2018, 70–72.
99 In the context of the philosophy of mathematics this stance is usually called Platonism.

Another aspect of the present work, which if not fictitious, is at least poetic, arises from the fact that higher-dimensional discourse exceeded the mathematical early on, and was subject to reinterpretations in fiction and the supernatural. These aspects are included here as the purpose of my artifacts is not just to act as models of the four-dimensional structures, but also to facilitate experiencing the subject matter in its poetic context.

My background in art education prompts me to embrace and also include these connotations of hyperspace that resonate with our imagination through their narrative power. Reviewing these aspects of higher space serves not only to inspire and contextualize the construction making, but to fully apprehend the fourth spatial dimension as it has been and still is understood, mediated, and applied. As Nevanlinna, writing on the truth effect of "works" in artistic research, puts it: "Anyone who does not want to accept this as a mode of truth is forced to shut virtually all human existence outside 'real' being."[100]. Varto also emphasizes the importance of inclusion, as artistic practice and research both rely on it to make their subjects appear in a multi-faceted manner, i.e. as a phenomenon[101].

The scope of the present study prevents me from giving anything more than a brief mentioning of these instances, and I refer my reader to the works of Blacklock[102], Volkert[103], and White[104] for the full account of this peculiar history.

IN FICTION LITERATURE

In works of modern fiction, the fourth spatial dimension usually acts as a plot device. Early science fiction in particular was influenced by the concept. H. G. Wells made use of higher space in his short stories, as in the story about a man who – as a result of a chemical explosion – is flung through hyperspace and rotated into his mirror image[105]. The concept of higher spatial dimensions also makes an appearance in the works of e.g. Fyodor Dostoevsky[106], Oscar Wilde[107],

100 Nevanlinna 2002, 65.
101 Varto 2018, 82–84.
102 Blacklock 2018.
103 Volkert 2018.
104 White 2018.
105 *The Plattner Story* (1896).
106 *Brothers Karamazov* (1880).
107 *The Canterville Ghost* (1887).

George MacDonald[108], Joseph Conrad[109], Madeleine L'Engle[110], and Jorge Luis Borges[111]. More recently, Thomas Pynchon's postmodernist novel *Against the Day* (2006) has been argued to have an extensive four-dimensional subtext[112].

The most geometrically advanced work is perhaps Rudy Rucker's 2002 novel *Spaceland*[113]. As a professional mathematician and a prolific science fiction author with extensive work also done in the popularization of four-dimensional geometry and its cultural history, Rucker draws an albeit fictional, yet topologically plausible account of hyperspatial beings coming in contact with our 3D world. Noting how e.g. in Wells' *Plattner Story* the intestines of the protagonist would actually fall out in hyperspace as the skin does not suffice to hold them inside it in all directions, Rucker resolves the issue by imbuing his hero Joe Cube with a 'hyperskin' covering the *ana* and *kata* facing parts of his body. Clifford A. Pickover has touched upon similar problems in his writings[114], focusing on the god-like powers of hyperbeings.

Science fiction short stories featuring four dimensionally enhanced objects have a special bearing on our pursuit here, e.g. in Algernon Blackwood's *A Victim of Higher Space*[115], the 'victim' keeps slipping to four-dimensional space – a condition brought about by his use of Hinton's cubes. Another instance, written by Henry Kuttner and Catherine L. Moore under the pen name of Lewis Padgett, is *Mimsy Were the Borogoves*[116], which portrays two children who after training manage to move along the fourth dimension of space. The knowledge required for the feat is mediated by Lewis Carroll's nonsense poem *Jabberwocky* and a foldable puzzle toy composed of interlocked wires and beads in the form of a tesseract. Finally, Walter Tevis's *The Ift of Oofth*[117] describes an unfolded model of a five-dimensional cube, which folds up and creates a portal extending through time and space. Although the models presented as the results of my research here

108 *Lilith* (1895).
109 *The Inheritors* (1901).
110 *A Wrinkle in Time* (1962).
111 *There Are More Things* (1975).
112 St. Clair 2011.
113 Rucker, Spaceland 2002.
114 Pickover 1999.
115 Blackwood 1914.
116 Padgett 1943.
117 Tevis 1957.

may not enable us to escape our three-dimensional space, they offer, however, an embodied approach to the study of four-dimensional geometry.

IN THE SUPERNATURAL

Toward the end of the 19th century, the fourth dimension of space attracted attention as a rational explanation to alleged occurrences of paranormal phenomena, especially spiritism, which were fascinating the public in the Western world. Although a precursor of this interpretation can already be seen in the 'essential spissitude' of spirit by 17th century theologian Henry More[118], the discussion on hyperspace was hijacked by occultists when German astrophysicist Friedrich Zöllner declared the famous medium Henry Slade as being capable of manipulating objects in four dimensions. In his *Transcendental Physics*, published in 1880, he described the elaborate experiments he conducted to prove Slade's feats as manifestations of actual four-dimensional reality. Slade was to transform the molecular structure of tartaric acid[119], and insert and remove coins and chalks from closed boxes[120]. As Zöllner had learned from Felix Klein the mathematical fact that all knots are actually unknotted in four-dimensional space, he had Slade untangling a knotted rope that had its ends sealed together with a piece of wax[121]. Slade could also see objects hidden from his view, as his soul would rise above our three-dimensional slice so the he could see, or at other times beings residing in the higher space gave him the information[122].

Klein later lamented Zöllner's misinterpretation of the mathematical fact into physical reality[123], and as a consequence of the Zöllner-Slade affair, the scientific credibility of four-dimensional geometry as a whole was somewhat tarnished in public opinion. Among the followers of the spiritualist movement, the reception of the fourth dimension as an explanation for psychic phenomena, Zöllner's results in particular, was highly differentiated[124]. Nonetheless, his work was

118 *The Immortality of the Soul, so farre forth as it is demonstrable from the Knowledge of Nature and the Light of Reason* (1659).
119 Blacklock 2018, 52.
120 Blacklock 2018, 59.
121 For speculations on different methods behind Slade's illusion, see Gardner's *The Church of the Fourth Dimension* (1991b).
122 Blacklock 2018, 60.
123 Klein 1979, 157–158.
124 Valente 2008.

cited favorably by e.g. Kandinsky and Jung[125], and Fechner attended two of Slade's séances, an activity said to have severely damaged his reputation as a scientist[126]. Along with Hinton's work, Zöllner was also an important inspiration to the esoteric writings of P. D. Ouspensky[127].

Hyperspatial rationale also received some support within the Theosophical Society. Although Helena Blavatsky herself refused the idea of four-dimensional space[128], her pupil C. W. Leadbeater refers to Hinton's writings and equates his fourth dimension of space with the Neo-Platonist notion of "astral plane" in a lecture given before the Amsterdam Lodge in April 1900[129]. To Leadbeater astral sight afforded clairvoyance through solid objects and direct perception of four-dimensional structures, such as the tesseract[130]. Investigations into the astral world also involved Leadbeater experimenting with fellow occultist Annie Besant on the transmission of 'thought-forms', the illustrations of which e.g. by drawing, they claimed had fundamental obstacles corresponding to the difficulties in four-dimensional visualization[131]. Rudolf Steiner, the founder of anthroposophy, was also an enthusiastic advocate of the mystic access into the supernatural fourth dimension, working extensively on four-dimensional visualization exercises and borrowing methods from Hinton[132].

Within the Christian tradition – and in addition to the interpretation given by Henry More – an Anglican priest, Arthur Willink, entertained the notion of four-dimensional explanations to spirits of the departed; kenosis and resurrection of Christ; and the omnipotence, omnipresence, and omniscience of the God in his 1893 book *The World of the Unseen*[133]. Similar ideas were discussed in *Another world; or, The fourth dimension* by Alfred Schofield[134] and in W. F. Tyler's 1907 book *The Dimensional Idea as an Aid to Religion*[135]. Martin Gardner has suggested, albeit tongue-in-cheek, that the theologies of Karl Barth and Karl Heim have hyperspatial subtexts[136].

125 Blacklock 2018, 43.
126 Fellner 2011, 129.
127 Claude Bragdon translated Ouspensky's *Tertium Organum* in English.
128 Blacklock 2018, 139.
129 Leadbeater 1918, 4.
130 Blacklock 2018, 163.
131 Blacklock 2018, 161.
132 White 2018, 77.
133 Willink 1893.
134 Schofield 1888.
135 White 2018, 72.
136 Gardner 1991a.

All in all, the fourth spatial dimension has often served to give a logically and aesthetically pleasing explanation to various fantastic notions in the supernatural, such as clairvoyance, telekinesis, out-of-body experiences, and spiritual influence in the material world. Consequently the objects pursued in the following report take on the poetic mode of an occult, or perhaps even ritual artifact. Although for a mathematician the occult heyday of hyperspace might be an awkward topic, I fully embrace the esoteric history of the concept as it increases the poetic value of my work and enriches it with intriguing and even humorous intertextual references.

THE MODELS

MODEL 1: *The Kinochoron*

The subject of the first chapter is a kinetic, stereographic projection of the hexadecachoron. Its name, the *Kinochoron*, is a portmanteau of the words “kinetic” and “polychoron”. The symmetry of this model with its six hexagram links of wire form “Chinese crosses” at the eight vertices and permits a “hyper-rotation” of the structure. This model was first presented in the summer of 2014 at the annual Bridges conference on mathematical connections in art, music, architecture, education, and culture[137].

137 Luotoniemi 2014.

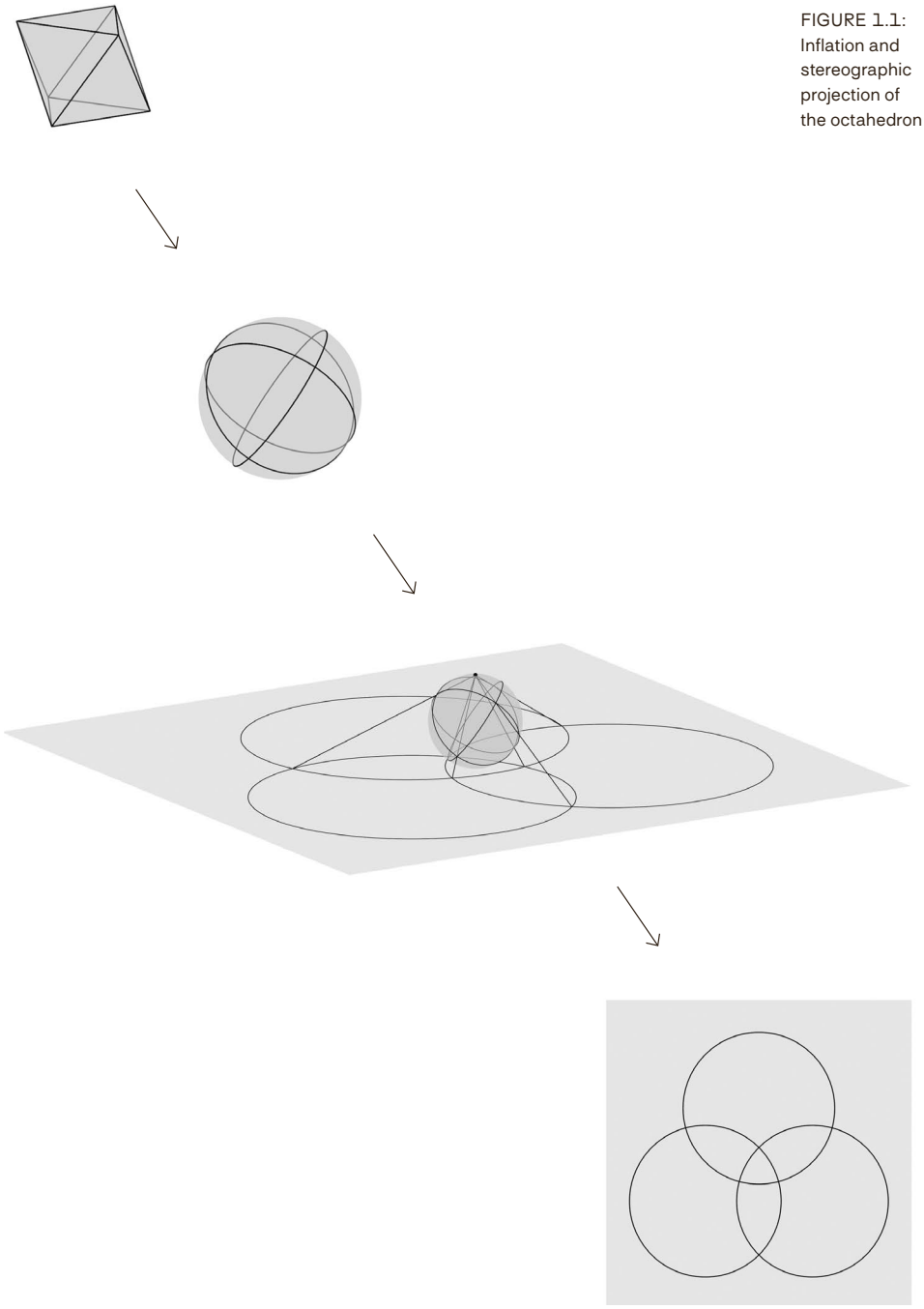


FIGURE 1.1:
Inflation and
stereographic
projection of
the octahedron

STEREOGRAPHIC PROJECTION

To prepare the polytope for stereographic projection, it is first inflated to a sphere. The stereographic projection then projects it from the north pole down to a picture plane tangent to the sphere at the south pole. Stereographic projection preserves the angles but bends the edges to curves. FIGURE 1.1 shows how the stereographic projection of the octahedron (six vertices, twelve edges, and eight triangular faces) produces three circles on the picture plane, intersecting at right angles at six points. If the octahedron is rotated under the projection, these circles will exchange their places.

The animated film *Dimensions*[138], which introduces four-dimensional polytopes with the help of dimensional analogy, cross-sections, and shadows, suggests that stereographic projection is the most efficient way to represent polychora because it does away with the inevitable overlap of the individual cells evident in parallel and perspective projections. Those projections preserve straight edges but distort the angles at the vertices, whereas stereographic projection preserves the angles but bends the edges to curves. When the polychoron under the projection is set into rotating motion in 4-space, the projection point travels through the cells and produces a mesmerizing visual effect of hyper-rotation, seen as an eversion of the cells in the stereographic projection output. Fritz Obermeyer's *Jenn3d* software even allows the user to rotate the polychora on their computer screen[139].

As a two-dimensional medium – like computer-generated animation, which lacks the tactile and plastic availability of a three-dimensional object – the stereographic projections of the polychora have also been implemented as 3D-printed models, most famously by Henry Segerman[140]. On the other hand, these rigid objects fail to communicate those structural relations of a polychoron that are further demonstrated in animations by the movement of the projection point with respect to the shape being projected. The topic of the paper at hand is the *Kinochoron* design – a stereographic projection of a hexadecachoron in a physical model that is capable of hyper-rotations evident in the animations of the same projection method.

Considering only the frameworks of the edges and excluding all two-dimensional faces, as is usually the case when

138 Alvarez, Ghys, and Leys 2008.
139 Obermeyer 2006.
140 Segerman 2016, 40–56.

designing physical models of the polychora, the realization of hyper-rotation in a three-dimensional object still seems – at first glance – to require stretching material to be used on the edges. This problem is eliminated by choosing to model the four-dimensional cross-polytope, the hexadecachoron, the cruciform vertices and consecutive edges which can be exploited to achieve a kinetic structure.

LINKING THE EDGES

The hexadecachoron has eight vertices, twenty-four edges, thirty-two triangular faces and sixteen tetrahedral cells. When the hexadecachoron is inflated to a hypersphere, and then stereographically projected into 3-space choosing the center of one of the cells as the projection point, the edges of the polychoron are arranged in six circles consisting of four edges each. This means that the edges can pass parts of their length to their neighbors belonging to the same circle through the vertices without changing the radius of the circle loop.

This results in some special requirements for a vertex, as it has to be designed so that the edge loops can move freely through it in three perpendicular directions without falling off the vertex completely. This is accomplished by duplicating and linking the edge loops appropriately, so that a vertex takes the form of a "Chinese cross" puzzle (FIGURE 1.2). Giving each of the vertex-to-vertex segments a 3/4 twist further reinforces the vertices and edges (FIGURE 1.3). Hence each two-strand edge loop twists three times in total and acquires the form of a hexagram link (FIGURE 1.4).

The edge loops are easily constructed from steel wire – a material suitable for the purpose of manufacturing kinetic models, as it gives in to substantial bending without losing its preference to straighten up. The final design, the *Kinochoron* (FIGURE 1.5), consists of steel wires encased in thin brass tubes. This manufacturing technique allows the use of butt joints that are essential for easy sliding of the edges.

HYPER–ROTATION

The completed design consists of twelve circular hoops. Each hoop forms a hexagram link with its mate, and a simple Hopf link with each of the ten hoops belonging to other hexagrams. Each of the six hexagrams comprises four edges of the hexadecachoron, resulting in a total of twenty-four. All of the hexagrams take part in

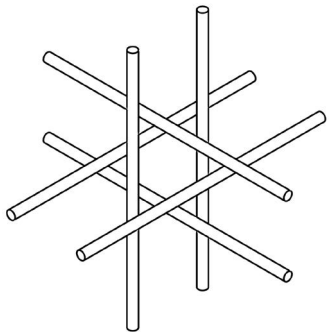


FIGURE 1.2:
A vertex, a
'Chinese cross'

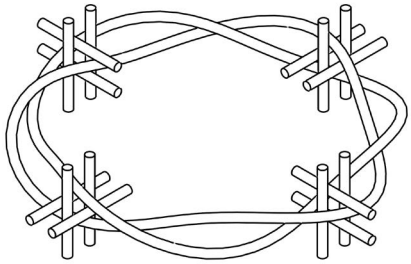


FIGURE 1.3:
3/4 twists on
four edges
between four
vertices on an
edge loop

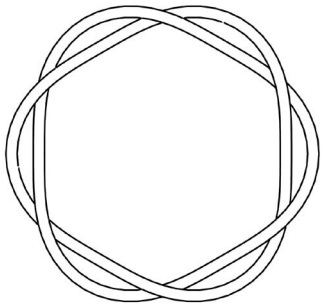


FIGURE 1.4:
A loop of
four edges,
a hexagram
link

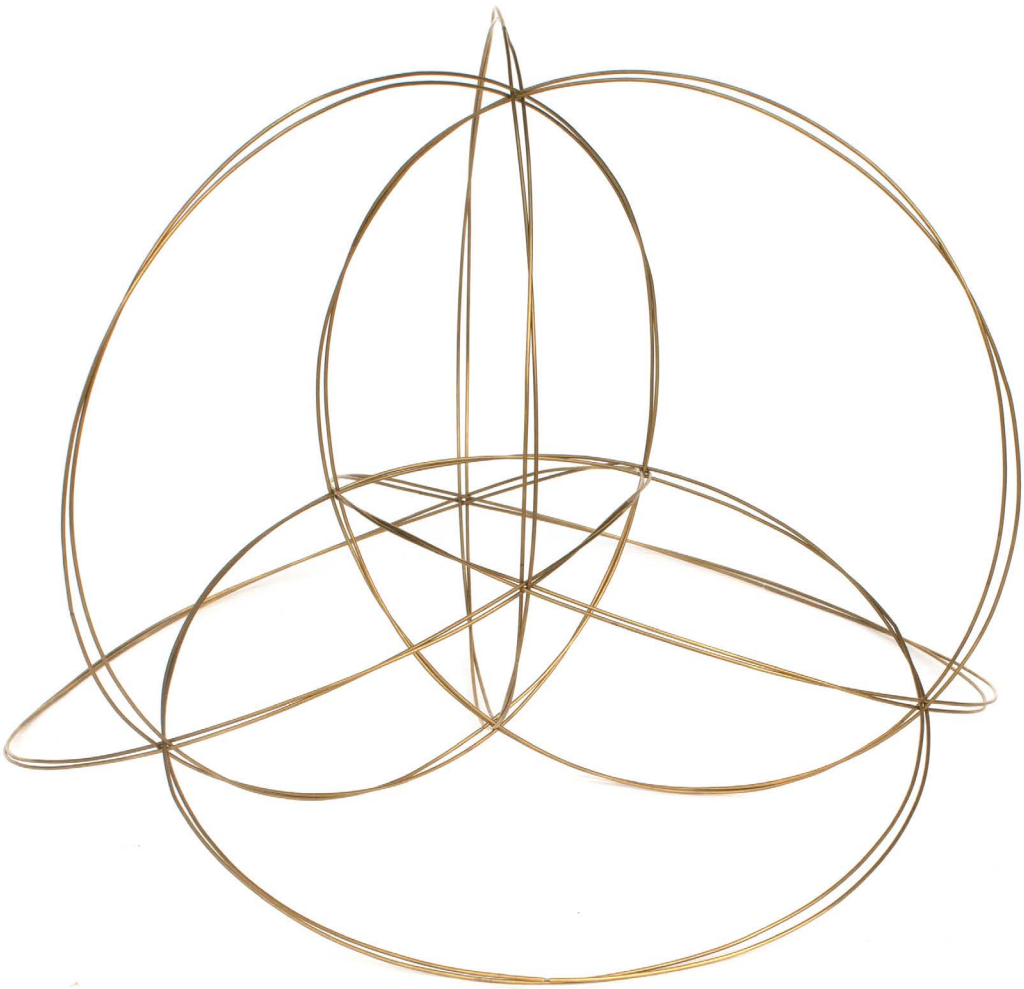


FIGURE 1.5: The *Kinochoron*

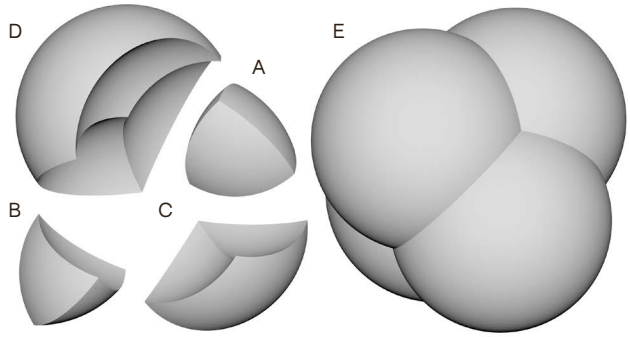


FIGURE 1.6: The five different types of tetrahedral cells in the stereographic projection of the hexadecachoron

defining each of the sixteen tetrahedral cells with one edge, and each edge takes part in defining four cells. All the hexagrams are joined together with eight "Chinese crosses", corresponding to the eight vertices of the hexadecachoron. At each vertex, six edges from three different hexagrams meet. In terms of knot theory, the entire design is a link of twelve components.

TYPE:	A	B	C	D	E
LAYER:	1	2	3	4	5
QUANTITY:	1	4	6	4	1
CONVEX FACES:	4	3	2	1	0
CONCAVE FACES:	0	1	2	3	4

The *Kinochoron* represents the edges of a stereographic projection of the hexadecachoron into 3-space where the projection point is the center of one of the cells. By examining the edges, we can imagine the spheres on which they lie and the spherical triangles they define. By combining these triangular faces, we can visualize the tetrahedral cells. In the *Kinochoron*, the tetrahedral cells come in five distinct types of shape (FIGURE 1.6). These tetrahedral types are aligned on five nested layers with varying quantities, and have convex and concave faces as follows:

The type E tetrahedron is a projection of the cell containing the projection point, and has its center projected at infinity. This causes the cell to be perceived as from the inside. In the entirety of the structure, a pair of tetrahedral cells can share a face, an edge

or a vertex. If the two tetrahedra are not connected by any of these zero-, one- or two-dimensional boundaries, they are situated on the opposite sides of the hexadecachoron.

As a three-dimensional object, the *Kinochoron* exhibits the two- and three-fold rotations of a tetrahedron that can be perceived just by turning the object. In addition to these symmetries, the kinetic interweave of the wires in the design allows a manipulation (FIGURE 1.7) that makes all the tetrahedral cells acquire new positions and shapes in the structure. This movement is analogous to the hyper-rotation in stereographic projection where the projection point follows the edges of the dual tesseract, i.e. the projection point travels from the center of a cell to the center of the neighboring cell through the center of the face they share. By pulling the opposite edges of a type C tetrahedron, it is possible to perform two cell-to-cell rotations with a single movement and turn a tetrahedron of type C into a tetrahedron of type E. While manipulating the model, we can imagine ourselves flying from cell to cell through the invisible triangular faces. The cells expand in turns fitting us and the entire surrounding universe inside them while the whole structure is observed in front of us as if seen through an extraordinarily powerful fisheye lens.

A stereographic projection of the hexadecachoron in which the projection point is in the center of a cell projects all the circles to same size – a feature which the *Kinochoron* portrays faithfully. It must be noted, however, that the unstable structure resulting from the sliding vertices usually depicts only the overall shape of the projection, not its specific edge lengths, unless the vertices are intentionally slid along the edges to adjust the correct edge lengths. Moreover, although the manipulation of the model resembles the precise hyper-rotation shown in computer-generated animations of stereographic projections, the hexagrams of the *Kinochoron* bend slightly to go around each other (see FIGURE 1.7, middle). In an accurate projection, the radii of the corresponding circles would vary slightly.

When all twelve hoops are pushed together, the model collapses flat for easy transport (FIGURE 1.8). Consequently it can be represented as a braid, an ornamental depiction of which is given in FIGURE 1.9. The Chinese crosses of the eight vertices are quite easy to spot on the picture, and with some effort it is possible to find all sixteen tetrahedral cells as well.

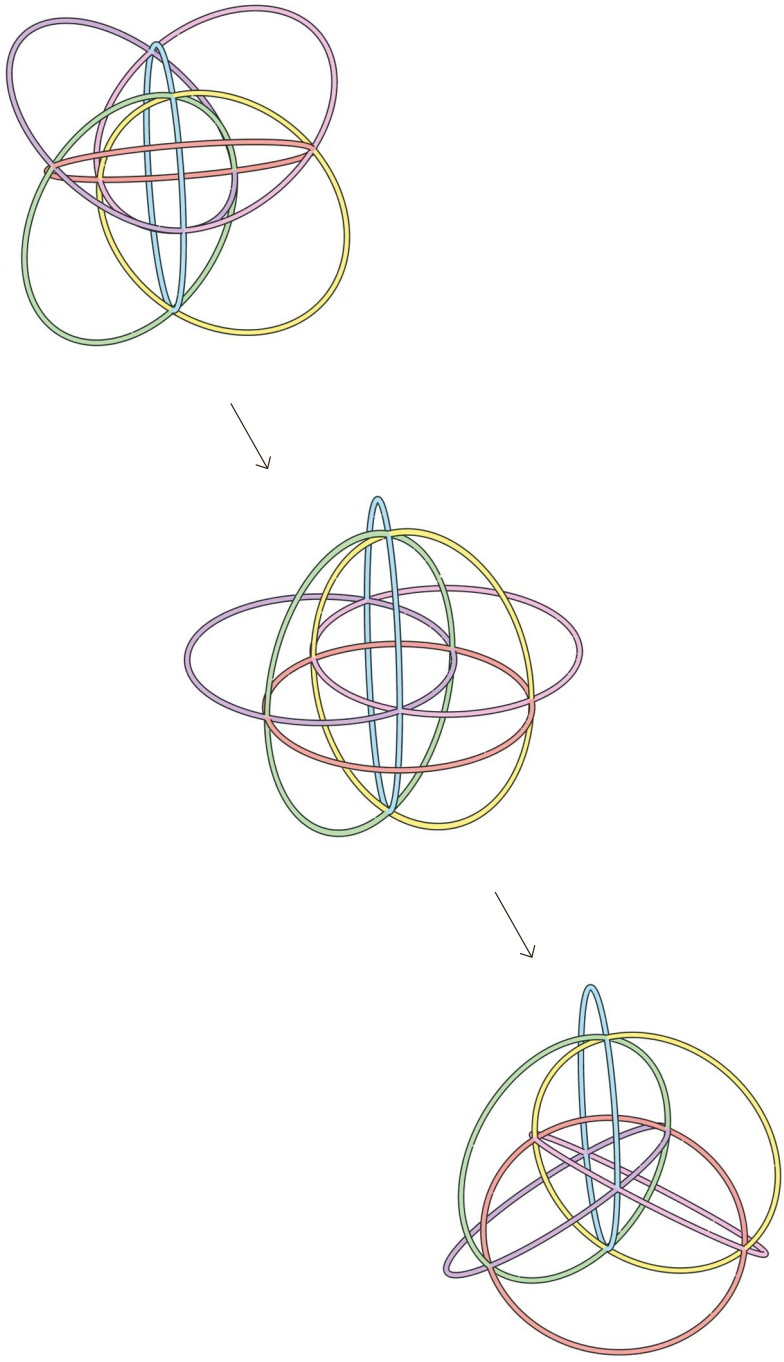


FIGURE 1.7: Cell-to-cell hyper-rotation of the *Kinochoron* through a face (with color coding on the hoops)

FIGURE 1.8:
The Kinochoron folded flat

CLOSING REMARKS

The *Kinochoron* is useful in understanding the structure of the hexadecachoron and demonstrating how on the surface of the hypersphere, as on the actual polychoron in 4-space, all of its sixteen tetrahedra are identical. What gives the *Kinochoron* its pleasant kinetic character is that it has just enough degrees of freedom for the hyper-rotation, but not more. The twists on the edges force the neighboring vertices and cells into movement, and all the parts of the portrayed hexadecachoron follow the manipulation faithfully to their designated configurations while preserving their respective relations.

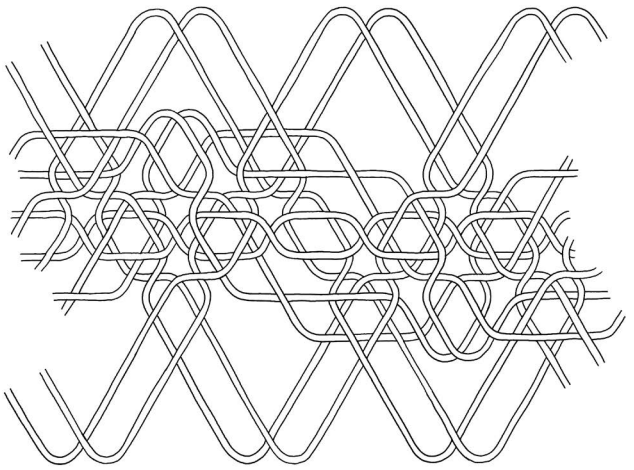


FIGURE 1.9:
The interweaved wires of the *Kinochoron* as a braid

Considering kinetic stereographic projections of other polychora, it is evident that alternative weavings have to be devised. One such solution for the pentachoron is presented in FIGURE 1.10. Unfortunately the friction at the wires makes the hyper-rotation quite laborious to achieve. The model *Wheels Within Wheels* (FIGURE 1.11) depicts a stereographic projection of the icositetrachoron, with twenty-four vertices, ninety-six edges, ninety-six triangular faces, and twenty-four octahedral cells. Having circles of two different radii prevents it from exhibiting any kinetic qualities. This model is interesting in its own right, as it demonstrates a surprising fact: how the sixteen wire loops can be interlaced so that it does not need any glue or ties at the vertices. A stereographic projection of the hexacosichoron could be crafted from interweaved wires as well, but the great differences in the edge lengths of such an intricate construction would arguably render the model structurally inconvenient.

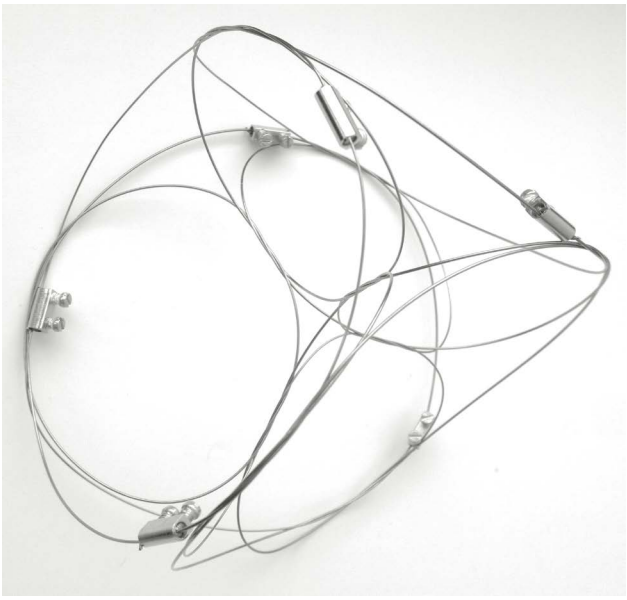


FIGURE 1.10: A hyper-rotating wire model of the 5-cell (truncated 5-cell)

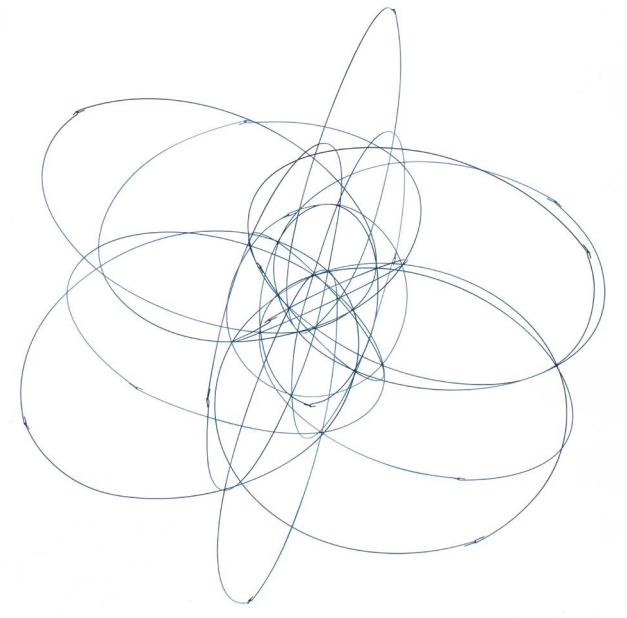


FIGURE 1.11: 'Wheels Within Wheels' – a stereographic projection of the 24-cell

MODEL 2: *Cube at Large*

The pair of models discussed in this chapter portray the tesseract in a *perspective projection* with four vanishing points. Because a setting like this is completely analogous to portraying a cube on a piece of paper as a perspective picture with three vanishing points, we will inspect that situation first to set up some vocabulary and make the relevant observations.

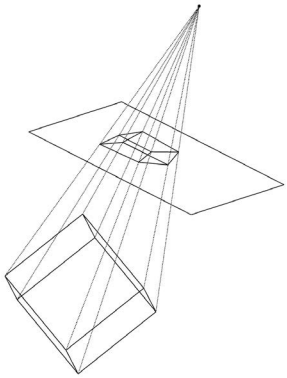


FIGURE 2.1:
The cube, the eye, and the picture plane

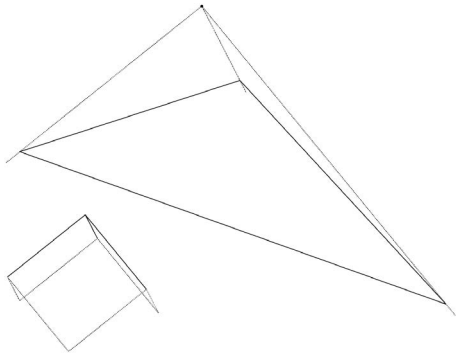


FIGURE 2.2:
The construction of the vanishing triangle

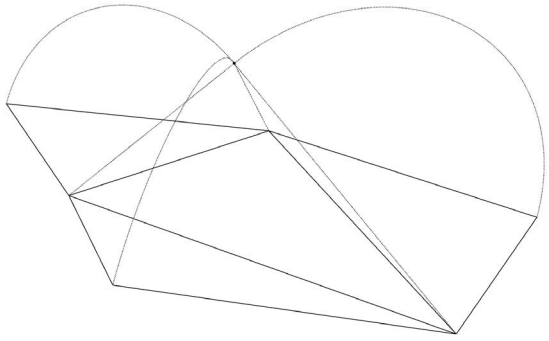


FIGURE 2.3:
Unfolding the pyramid walls

THREE–POINT PERSPECTIVE

A standard exercise in perspective drawing is to construct a projection of the cube using three vanishing points. There is lot of freedom in placing the vanishing points on the paper, but getting the proportions of the actual object right requires more care. The challenge is getting the representation to be that of a cube, the edges of which all have equal lengths, instead of just producing a projection of a box. In his *Topological Picturebook*[141], George Francis gives the instructions for the required constructions: the projection setting comprising the cube, the eye of the observer, and a picture plane.

An observer in 3-space is looking at the vertex of a cube – the three nearest faces meeting at that vertex being visible to them, and there is a picture plane between the cube and the eye of the observer, where we will trace the projection the observer sees (FIGURE 2.1). The vanishing points of the projection are found by translating the cube so that the aforementioned vertex meets the eye, and then extending the edges and faces meeting at that vertex until they intersect the picture plane (FIGURE 2.2). The intersection, called *the vanishing triangle*[142], has the vanishing points as its vertices, and the horizon lines as its edges. It is necessarily an acute triangle, as the corners of a triangle cut from a vertex of a cube must be smaller than right angles. An important insight is also that the walls of the pyramid emanating from the eye – having the shape of a right triangle, can be unfolded, i.e. rotated around the horizon lines until they lie in the picture plane (FIGURE 2.3).

Why does Francis's recipe for obtaining the vanishing points and horizon lines work? To understand the rationale we should consider what exactly a vanishing point is. Although it is customary to describe it as a point in the picture where parallel lines meet, it is not the most instructive approach to the concept. The vanishing point should, instead, be understood as *a line meeting the eye of the observer*, just *seen* as a point because fully foreshortened. Thus for every family of parallel lines there is a unique line that goes through the eye. This line is the vanishing 'point' of that family of lines. The superiority of this definition is that it generalizes naturally not only to higher dimensions (e.g. the horizon line of a family of parallel planes in the

141 Francis, A Topological Picturebook 2007.

142 Francis calls this the framing triangle: I prefer *vanishing triangle* as it is a figure made up from the vanishing elements.

scene is a unique plane meeting the eye of the observer), but also to the lower one[143]. With such a conception of vanishing elements, it is evident that the pyramid described above indeed intersects the picture plane in a vanishing triangle with the correct vanishing points and horizon lines.

Although the vanishing points afford the construction of a picture portraying a box bounded by six quadrilateral faces, the box is not yet guaranteed to be an actual cube. To ensure that the portrayed box really has square faces, Francis goes on to define the required *diagonal vanishing points*, which are found by fitting the corners of the unfolded walls with small squares. The diagonals of the small squares are extended until they intersect the vanishing triangle, determining the diagonal vanishing points (FIGURE 2.4). Once one edge of the portrayed cube is drawn (FIGURE 2.5), the rest of the cube can be completed by making sure the diagonals of its faces meet their horizon lines at the diagonal vanishing points (FIGURE 2.6).

Because the walls of the pyramid were unfolded into the picture plane, it is obvious that a perspective construction of the cube can be fashioned entirely in the plane. This entails setting up an arbitrary acute triangle, using Thales' theorem to construct the unfolded walls, then constructing diagonal vanishing points on the horizon lines edges, and finally drawing a legitimate perspective picture of a cube starting from one initial edge. As such, Francis's formulation suggests a higher-dimensional generalization – a model of the tesseract in four-point perspective, constructed entirely in our ordinary three-dimensional space.

FOUR–POINT PERSPECTIVE

To set up the scene for the construction of the four-dimensional version of the perspective picture described above, we imagine a four-dimensional being observing a tesseract in 4-space. The 'paper' on which the tesseract is projected is now a three-dimensional space like our own. Furthermore, we imagine translating the tesseract vertex nearest to the hyperbeing to its eye, and extending the four cells meeting at that vertex until they intersect the picture space.

143 A flatlander's perspective picture of a *square* is just *four points on a line*, and although in the picture line we cannot trace where the two pairs of parallel sides of the square would meet, the two vanishing points can be found with a method analogous to Francis's construction.

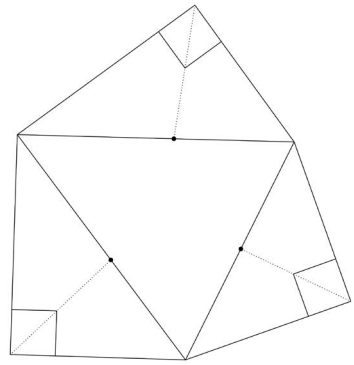


FIGURE 2.4:
The construc-
tion of the
diagonal

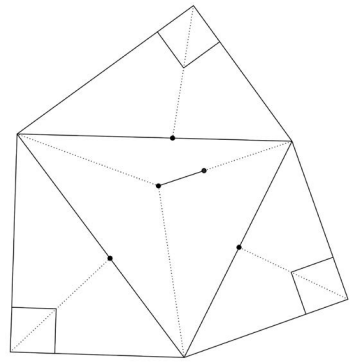


FIGURE 2.5:
First edge of
the cube

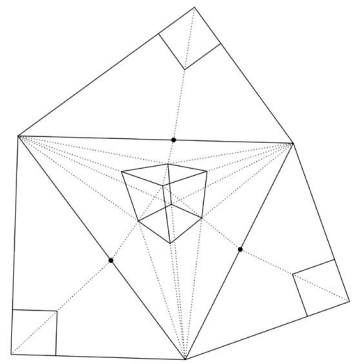


FIGURE 2.6:
Completing the
cube vanishing
points

The intersection yields *the vanishing tetrahedron*, comprised of four vanishing points, six horizon lines, and four *horizon planes*.

To build this vanishing tetrahedron, we must realize that it has to be both *acute* and *orthogonal* (FIGURE 2.7). As the vanishing tetrahedron is a slice cut from the vertex of a tesseract, it has to be acute where the cells meet at right angles at the faces. This means that the dihedral angles of the tetrahedron must be less than right angles. Also, the opposite pair of face planes meeting at a vertex of the tesseract but not sharing an edge are *absolutely perpendicular*[144], and thus the opposite edges of the tetrahedron acquired by slicing the tesseract around the vertex must be perpendicular, as they correspond to the absolutely perpendicular faces. Such tetrahedrons are called orthogonal[145].

The next step is to construct the four unfolded "walls" of the four-dimensional hyperpyramid suspended from the eye and based on the vanishing tetrahedron. These walls are three-dimensional pyramids based on the faces of the vanishing tetrahedron, with right angles at each apex. To erect such a pyramid, we construct a sphere on each edge, centered at the midpoint of the edge and going through the vertices of the edge (FIGURE 2.8). By Thales's theorem, any point on this sphere will suspend a right angle with the edge, and the intersections of the spheres will determine a point above[146] a triangular face suspending a right triangle with each of the edges (FIGURE 2.9). The diagonal vanishing points are constructed analogously to the method used above by fitting the corner of the unfolded wall now with a small cube, and then finding the intersections of the extensions of the face diagonals with the horizon lines (FIGURE 2.10).

When this construction is done for all four triangular faces, all edges of the vanishing tetrahedron are fitted with diagonal

144 Unlike in our three-dimensional space, where planes usually intersect on a line, planes in four-dimensional space usually intersect at a single point. Furthermore, two planes are said to be *absolutely perpendicular* when every line lying in one of the planes through the intersection point is perpendicular to every line of the other plane through the same point. (Manning 1956, 81).

145 One way to construct an orthogonal tetrahedron is to start with a rhombohedron, and take alternating vertices to build a tetrahedron, guaranteed to be orthogonal.

146 There are two possible choices for each apex, one on the inside and one on the outside of the vanishing tetrahedron. Geometrically it does not matter which one of these we use, but to avoid cluttered figures, we use those sitting outside the vanishing tetrahedron.

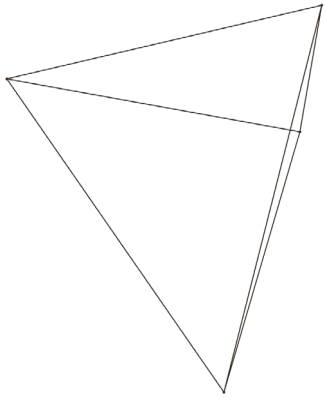


FIGURE 2.7: An acute and orthogonal tetrahedron

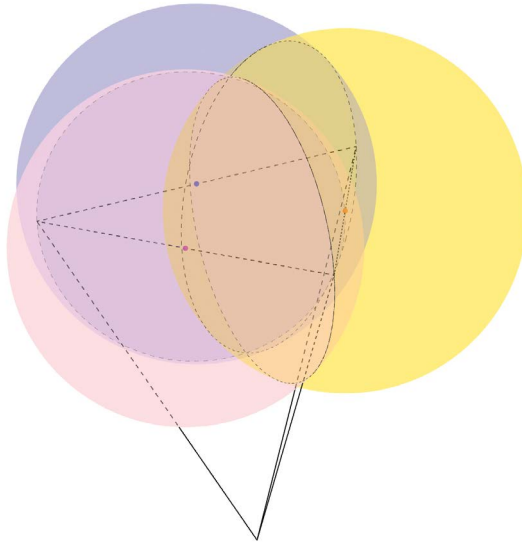


FIGURE 2.8: The spheres on the edges

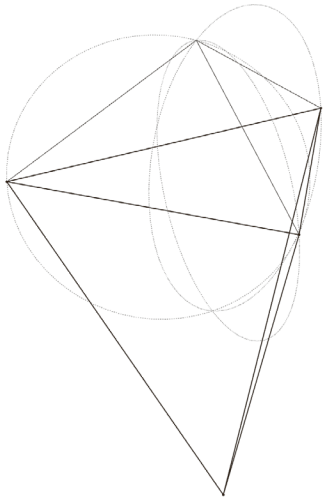


FIGURE 2.9: Construction of a hyperpyramid wall

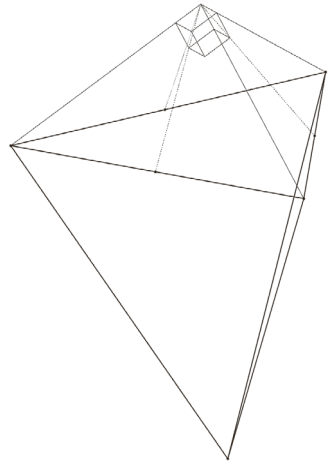


FIGURE 2.10: Construction of the diagonal vanishing points

vanishing points[147]. As the neighboring faces of the hyperpyramid walls are congruent, the diagonal vanishing points coincide (FIGURE 2.11).

With the vanishing points and the diagonal vanishing points in place, we are ready to start building the actual projection by placing the vertex of the tesseract closest to the eye of the four-dimensional observer (FIGURE 2.12). The first vertex could also be outside the vanishing tetrahedron, but as we are planning to suspend the projection by wires from the vanishing points in the physical model, we place the first vertex inside the tetrahedron.

The next step is to build one full edge protruding from the first vertex. This is done by placing a second vertex to one of the four line segments connecting the first vertex to the vanishing points (FIGURE 2.13). As one full edge is now determined, the artistic license expires. The edge length is translated on an adjacent edge line by the use of diagonal vanishing point (FIGURE 2.14 and FIGURE 2.15), and one square face is constructed (FIGURE 2.16). Continuing around the vertex we can construct all square faces protruding from it (FIGURE 2.17).

Connecting the outmost corners of these squares to the vanishing points, we can complete the nearest cubical cells (FIGURE 2.18), and connecting the outmost corners of these cubes again to the vanishing points, the farthest vertex of the tesseract is found (FIGURE 2.19) and the projection is complete.

The problem with this method is that it is difficult to predict how the projection will turn out from the few initial constructions. For the purpose of finding a pleasant composition where all the parts have enough space around them, Daan Michiels, a graduate student at the University of Illinois at Urbana-Champaign, wrote a python script that works as a plugin for Blender. The script constructs the four-dimensional Cartesian coordinates of the tesseract by a method of extrusion, and then uses a projection matrix to get the three-dimensional coordinates of the projection. The direction of view can be rotated via isoclinic quaternions[148].

147 In addition to the diagonal vanishing points on the edges, one can also construct the *space-diagonal vanishing points* on the horizon planes by extending the space diagonals of the small cubes, but it is not necessary for the construction of the tesseract model described here.

148 Michiels 2017.

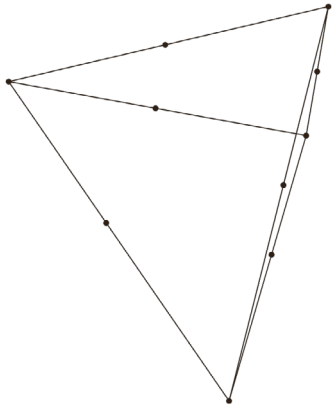


FIGURE 2.11

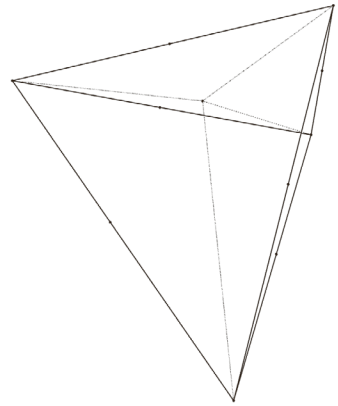


FIGURE 2.12

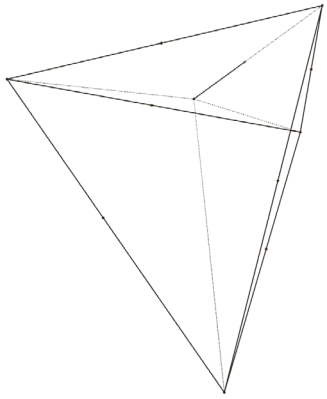


FIGURE 2.13

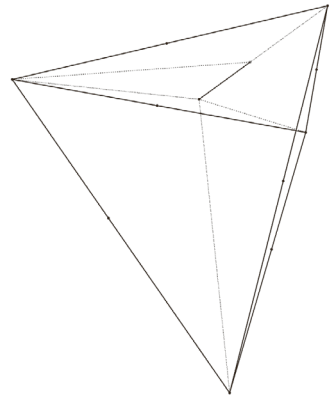


FIGURE 2.14

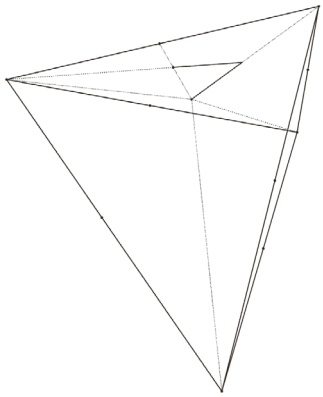


FIGURE 2.15

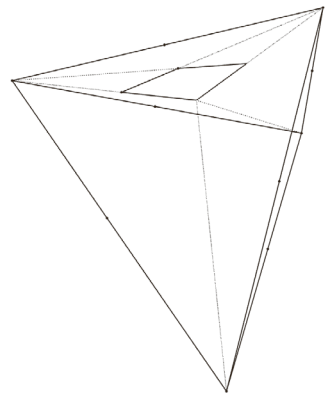


FIGURE 2.16

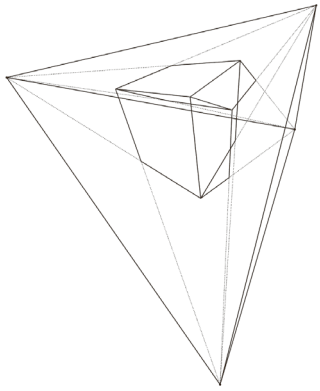


FIGURE 2.17

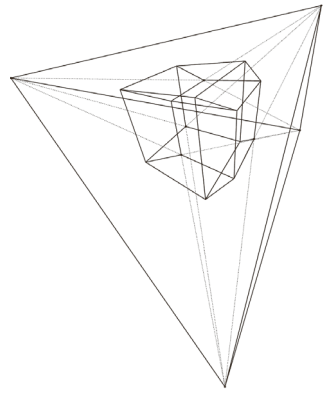
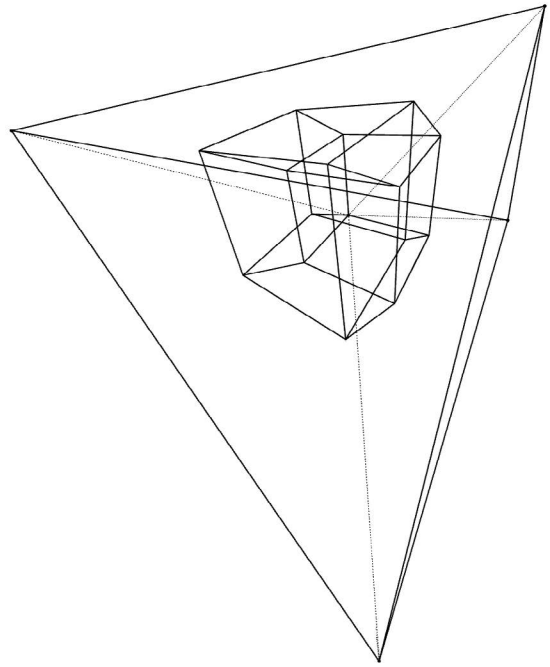


FIGURE 2.18

FIGURE 2.19



The physical model, *Breadth & Length & Depth & Height*[149], portrays the projection with brass tubes. The wires suspending the model inside the vanishing tetrahedron of steel show how parallel edges converge to four vanishing points (FIGURE 2.20). An illustration resembling a tesseract in four-point perspective was previously constructed by Victor Schlegel[150] for his 1882 article[151], although he had one of the vanishing points projected to the center of a cubical cell, resulting in the conventional cube-in-a-cube composition.

149 *that ye, being rooted and grounded in love, May be able to comprehend with all saints what is the breadth, and length, and depth, and height* (Ephesians 3. 17–18).

150 Schlegel is better know for the *Schlegel diagrams*, which are perspective views of polytopes seen from the center of one of their facets.

151 Schlegel 1882, 195.

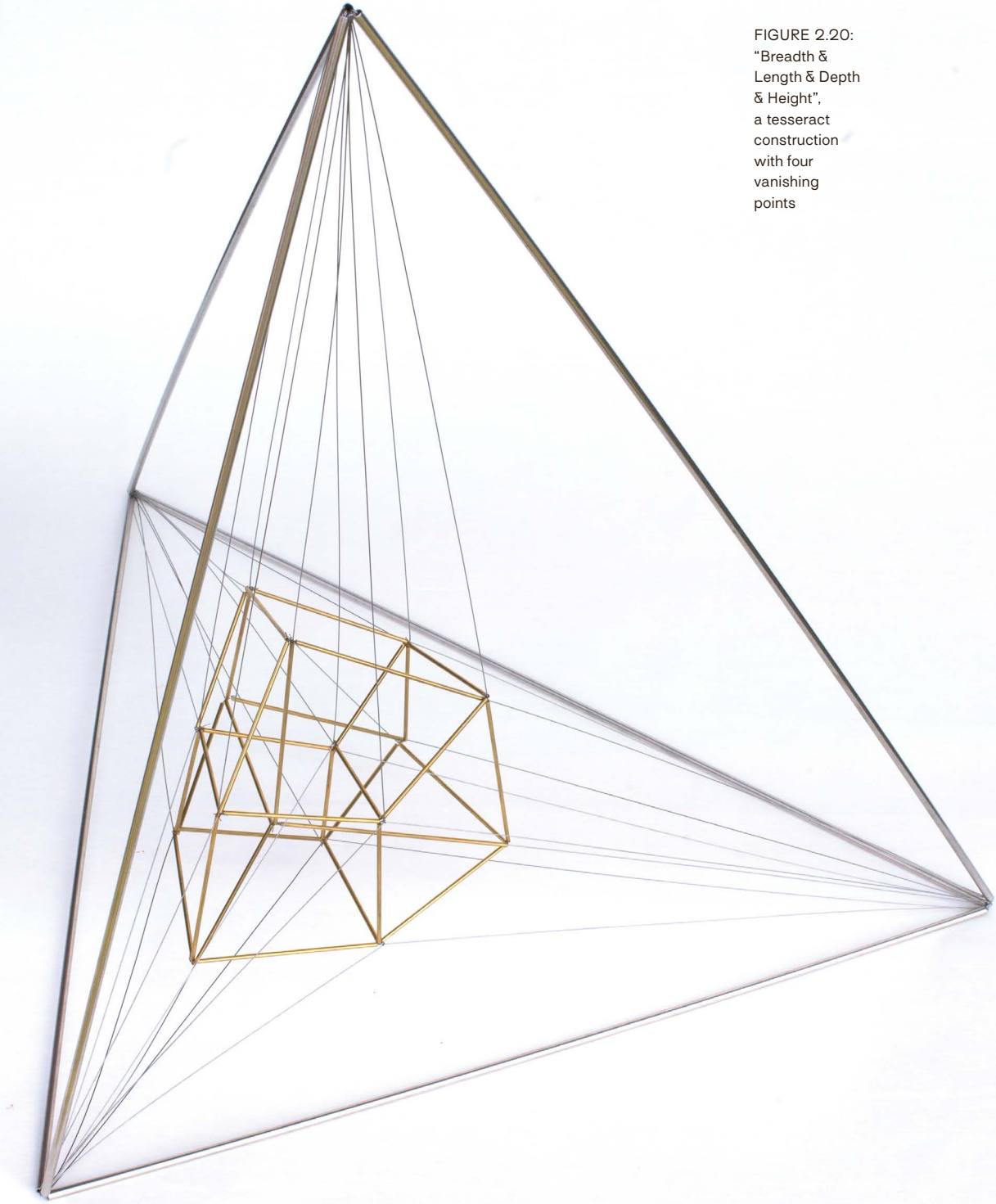


FIGURE 2.20: "Breadth & Length & Depth & Height", a tesseract construction with four vanishing points

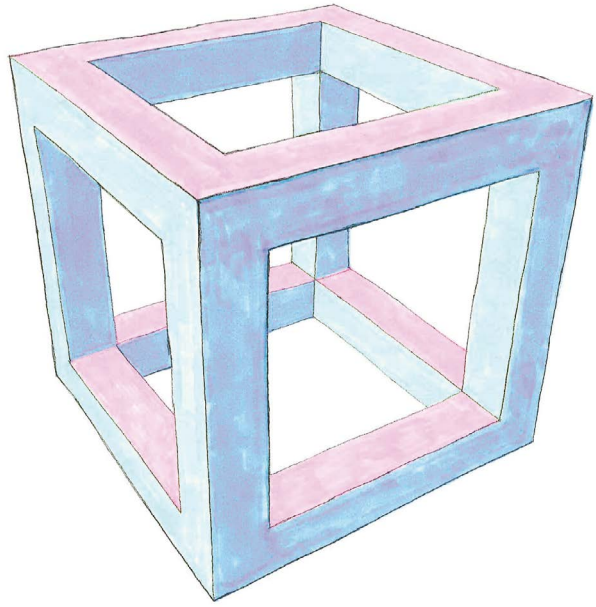


FIGURE 2.21: Illustration of the cube with solidified and colored edge frame

SOLIDIFYING THE EDGES

To further emphasize the expression of depth – or *hyper*-depth, rather, we can solidify the edges of the tesseract to show how they taper toward the vanishing points under the perspective projection. Before doing this, let us first look at how this effect is achieved in the lower-dimensional case.

In planar illustrations of polyhedra it has been customary since Leonardo[152] to portray the solids as frameworks of edges appropriately thickened. The thickness of the edges serves the cause of depth perception by conveying the information of how far away from the eye of the observer each part of the object is, the bulkier parts being in front and the slimmer ones at the back. The solidifying of the frame affords the conveying of additional information about the actual object, and coloring of the regions can be used to tell which directions the facets are facing toward in the original 3-space (FIGURE 2.21). The constructions needed are simple: first each of the eight vertices of the cube is fitted with a smaller cube, and then the collinear edges of neighboring cubes are connected by line segments.

152 Illustrations in Luca Pacioli's *De Divina Proportione* (1509).

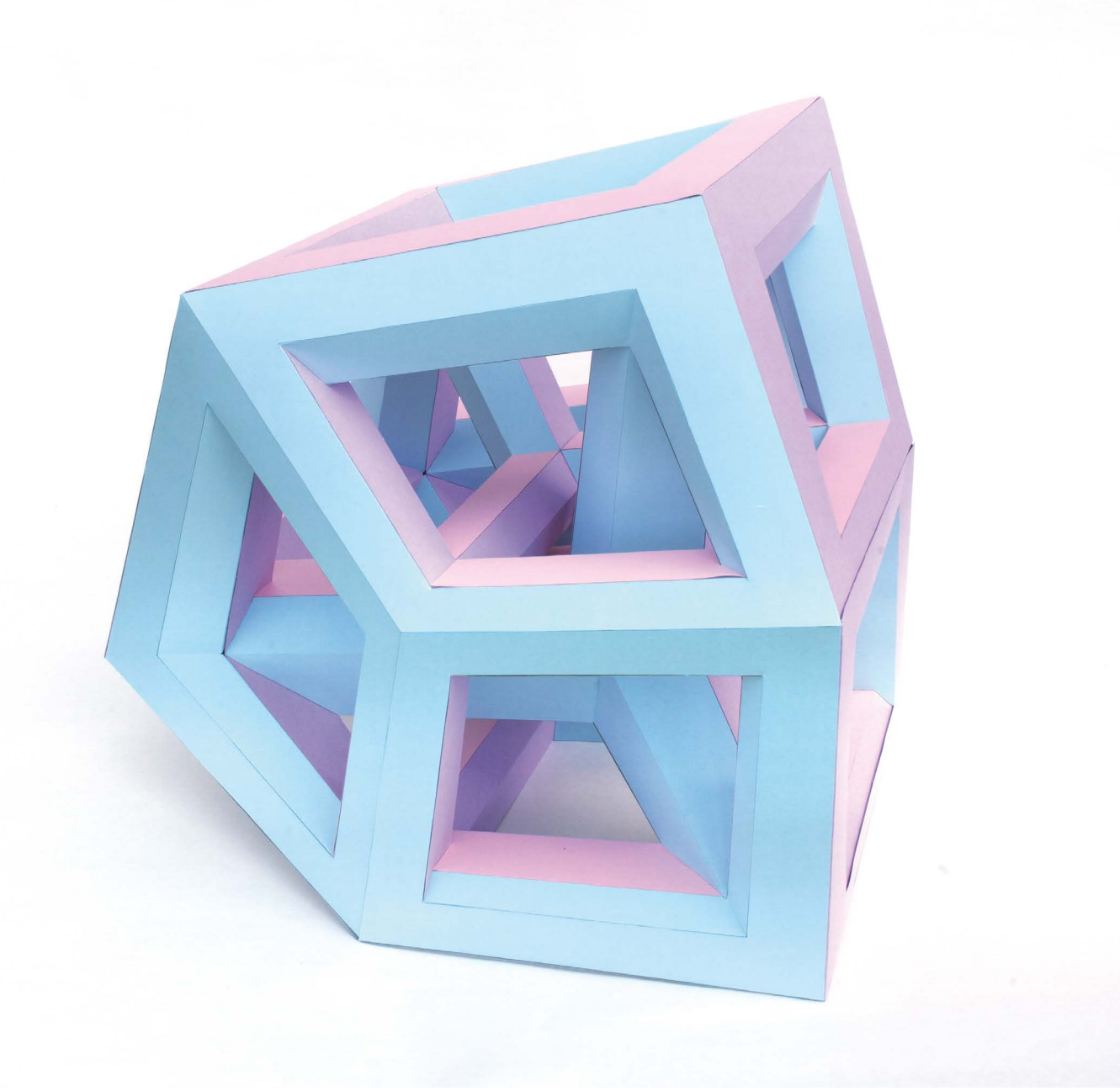


FIGURE 2.22: *Cube at Large*, a solidified frame of the tesseract in four-point perspective

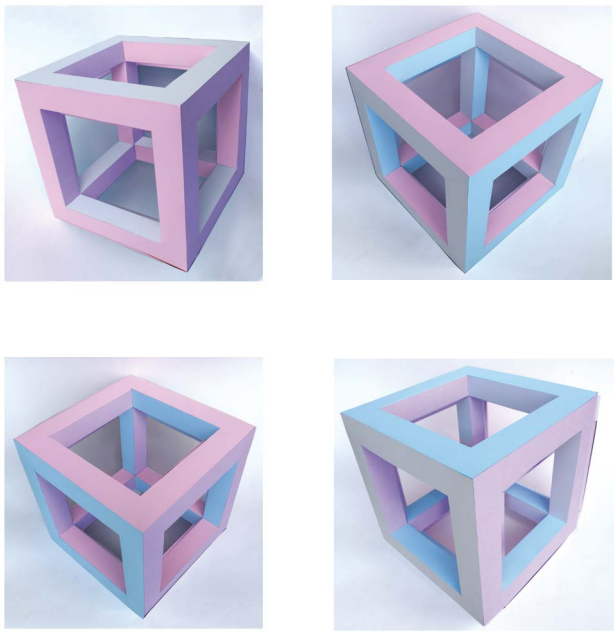


FIGURE 2.23: Views of the model from each of the four vanishing points

Following the lower-dimensional example, each of the sixteen vertices of the tesseract is fitted with smaller tesseracts, and then collinear edges of neighboring tesseracts are connected by line segments[153]. The cubical cells (or rather the sections of the prismatic beams belonging to the same cube) are colored so that the cubes facing the same perpendicular direction have the same color[154] (FIGURE 2.22).

The blueprints needed for the physical paper model were exported from the 3D modeled design in *Blender* with a plugin, which unfolded the mesh to planar vector graphic files and also equipped the facets with glue tabs[155].

Observing the model, a curious phenomenon is witnessed which one would not have thought about beforehand. If the model is viewed exactly from one of the four vanishing points, the

153 This method was also used by Scott Kim in his fourdimensional version of the impossible tribar (Kim 1978).

154 Instead of dark and saturated hues, I decided to use pastel colors, as their calm and pale appearance does not disturb the perception of those depth cues which are given by the play of light and shadow on the surface of the three-dimensional shape.

155 Dominec 2010.

edges pointing toward that point are completely foreshortened. Consequently the object will appear from these vantage points as a regular three-dimensional cube. (FIGURE 2.23)

CLOSING REMARKS

The pair of models – the perspective constructions of the tesseract – enables us to see the more familiar shapes of cube and square in the hierarchy of hypercubes, and also envision the higher rungs of the ladder. Unlike the other models described here, which are essentially just different unwrappings of the polychora's three-dimensional boundary into our 3-space, the perspective constructions might also give us an inkling of the actual hyperspatial volume bounded by the polyhedra. When compared to the stereographic projection, however, the perspective projection has a downside of making things overlap each other.

Future improvements on the *Cube at Large* could also be implemented with atmospheric perspective through the use of printed color gradients on the cardboard pieces. The correct fading of the hues should of course be calculated numerically with a script written for this purpose.

MODEL 3: *Crooked Houses*

Robert A. Heinlein's 1941 science fiction short story "... And He Built a Crooked House"[156] introduces a house built in the shape of a three-dimensional unfolding of a tesseract. As a result of an earthquake, the house folds 'up' to an actual four-dimensional tesseract, and the inhabitants are trapped inside. Like ants walking around the surface of a cube from square to square, they navigate the eight cubical rooms of the house taking straight-line routes northeast to southwest, southeast to northwest, up and down, only to find themselves back were they started after a round trip. Although such an experience would arguably be most unnerving, the chance to inspect the spatial interrelations of the cells of a regular polychoron would provide one with a more involved insight of four-dimensional space. This chapter describes a method of facilitating such investigations with hand-held kinetic models that employ hyperbolic tilings – patchwork surfaces composed of cloth hexagons based on the *bitruncated* versions of the tesseract and the pentachoron. These models were first presented in summer 2017 at the annual Bridges conference on mathematical connections in art, music, architecture, education, and culture[157].

156 Heinlein 1941.
157 Luotoniemi 2017.

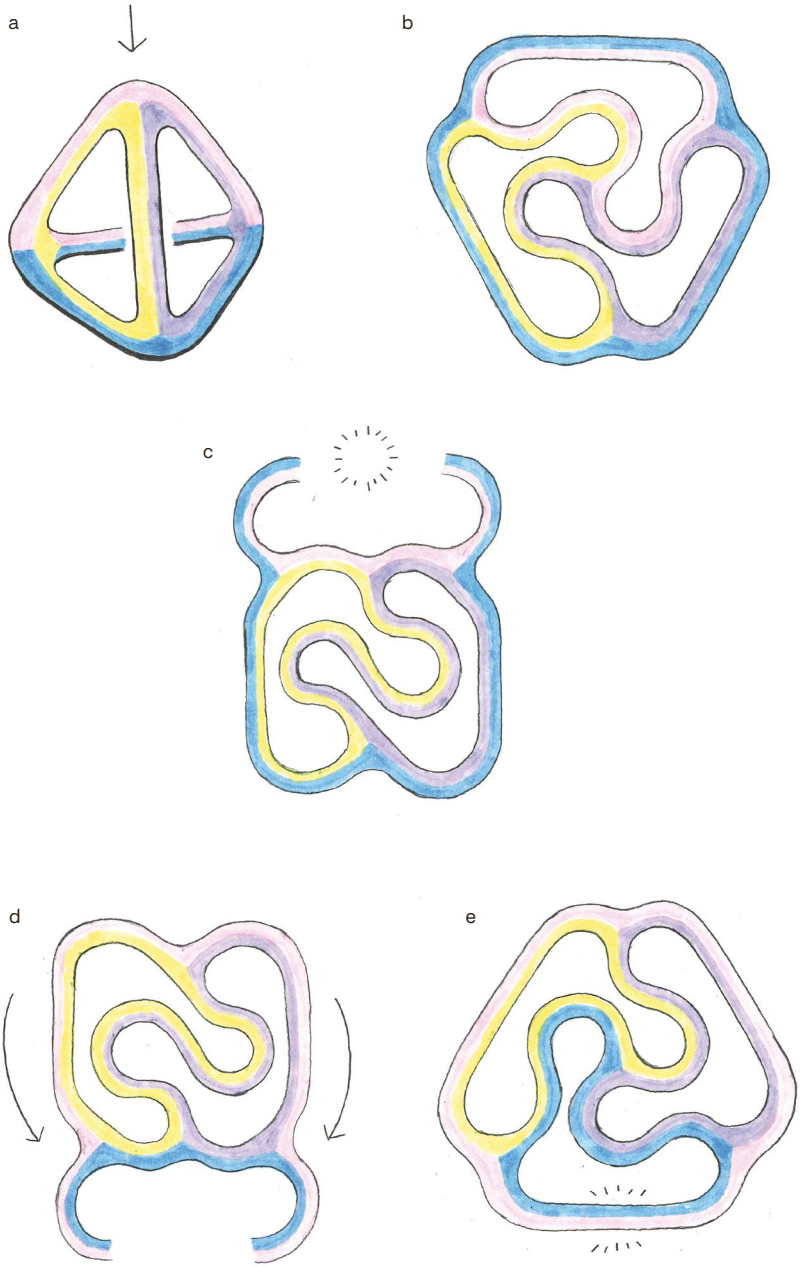


FIGURE 3.1:
Flexible tetra-
hedron model
in a plane.

SQUISHING A TETRAHEDRON INTO A PLANE

Let us first imagine a tetrahedron made from a soft material, like playdough (FIGURE 3.1a). If it were lying on a plane, the flatlanders inhabiting that plane would experience only the blue facet of the tetrahedron, the other sticking out to our three-dimensional space – invisible and fictitious to the flatlanders. If the tetrahedron is then squished into the plane, the edges previously sticking out would become wrinkled inside the triangular face initially in the plane (FIGURE 3.1b). The flatlanders could then puncture one of the soft edges to gain access to another facet, e.g. the pink one (FIGURE 3.1c). They could then move the loose ends of the soft edges to another side of the structure (FIGURE 3.1d). If they now connected the ends to make that edge whole again (FIGURE 3.1e), they would see that the structure would appear just like before in FIGURE 3.1b, only with the facets having changed their places. In the three-dimensional situation, this would correspond to tipping the tetrahedron over on the plane to stand on another of its faces.

This kind of educational toy for the flatlanders would perhaps be a bit unwieldy because of the constant cutting and connecting involved. In two-dimensional space objects are either connected or disconnected, and the existence of tunnels (openings that allow passage from one compartment to another, but still staying intact by their frame) is impossible. Fortunately, it turns out the existence of tunnel-like shapes in three-dimensional space affords a more elegant implementation of the idea, although achieving this goal turns out to require a special preparatory treatment for the polychoron, called a *truncation*.

TRUNCATED POLYHEDRA

Besides Platonic solids, there is a category of less regular objects called Archimedean solids. The vertices of these objects are identical, but the faces can be different. Some of these objects can be acquired through a method of truncation[158], which means cutting away the vertices with a plane perpendicular to a line defined by the center of the solid and the vertex in question at some depth. Original vertices will now be replaced with cross-sections as new faces. This

158 The word "truncation" is due to Kepler's naming of the Archimedean solids.

procedure shows many interesting relations between regular solids (most of them not discussed here), and can be generalized to work on polychora as well. Truncation is one of the interrelations of the polyhedra shown in an interactive app *Move & Turn*[159], invented by Jouko Koskinen and implemented as an app by Jeff Weeks.

The truncation – as discussed here – refers to the process of cutting away every vertex of a regular solid with a plane perpendicular to a line going through the center of the solid and the vertex. As a result, the original vertices are replaced by polygons whose shape is determined by the vertex figure of the original solid. The depth of the truncation is a free parameter. FIGURE 3.2 shows the truncation of a cube all the way to the dual octahedron, and the Archimedean solids met along the way. Notice how in the *truncated octahedron* the faces resulting from the truncation are already truncating each other. This stage is called the *bitruncation* of the polytope.

FIGURE 3.3 shows the truncation of a tetrahedron all the way to the dual tetrahedron, and the Archimedean solids met along the way.

TRUNCATED POLYCHORA

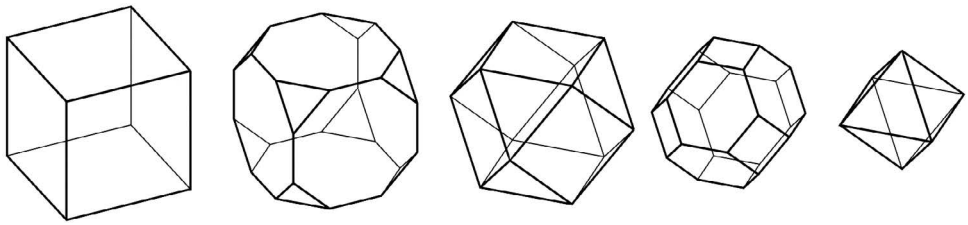
The four-dimensional regular polychora are truncated by cutting away every vertex of the polychoron with a three-dimensional space perpendicular to a line going through the center of the polychoron and the vertex. As a result, the original vertices are replaced by polyhedral cells whose shape is determined by the vertex figure of the original polychoron. The depth of the truncation is a free parameter. FIGURE 3.4 shows a sequence of truncations from the tesseract to varying depths, with the new polyhedral cells highlighted in gray.

BITRUNCATED TESSERACT

For the purpose of the patchwork surface model described here, we truncate the tesseract all the way to its *bitruncated* form (bottom right in FIGURE 3.4), where the cells resulting from the truncation have started to truncate each other. The bitruncated tesseract is composed of ninety-six vertices, 192 edges, thirty-two triangular faces, twenty-four square faces, sixty-four hexagonal faces,

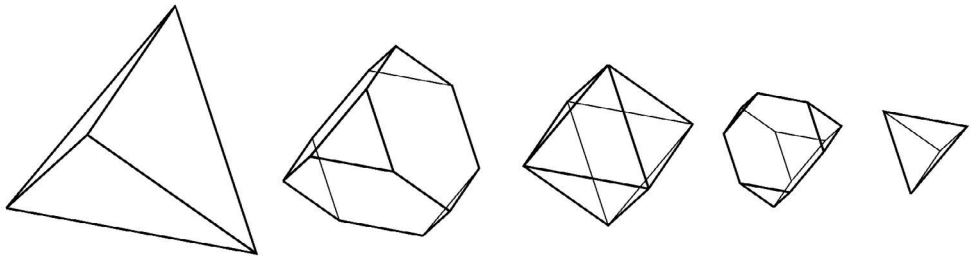
159 Koskinen and Weeks 2015.

FIGURE 3.2:



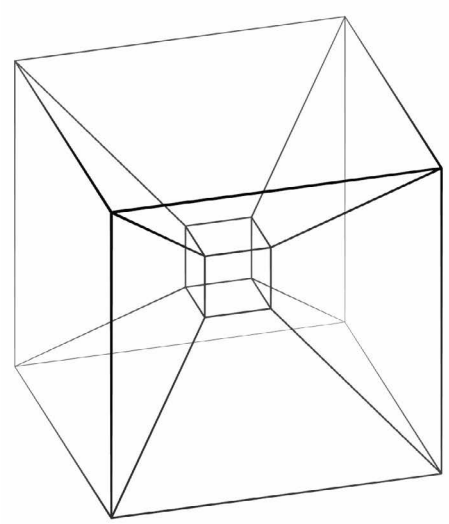
Cube | Truncated cube | Cuboctahedron | Truncated octahedron (bitruncated cube) | Octahedron

FIGURE 3.3:

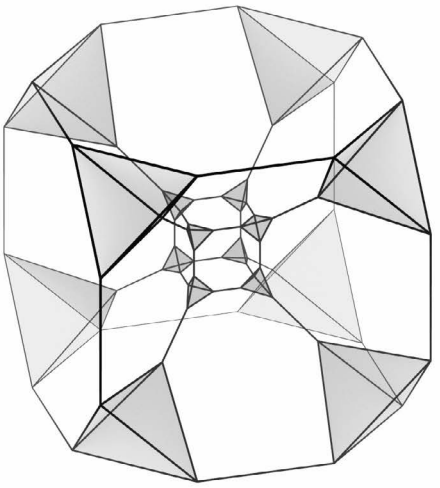


Tetrahedron | Truncated tetrahedron | Octahedron | Truncated tetrahedron (bitruncated tetrahedron), | Tetrahedron

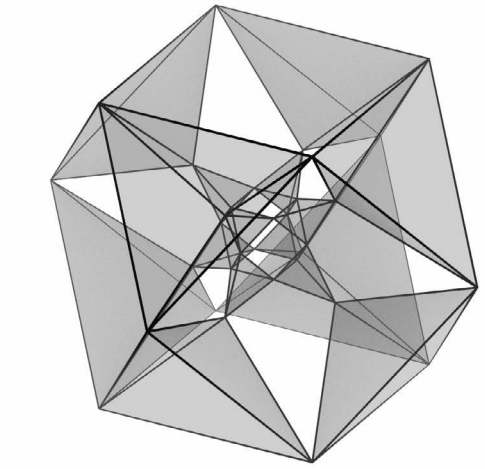
FIGURE 3.4:



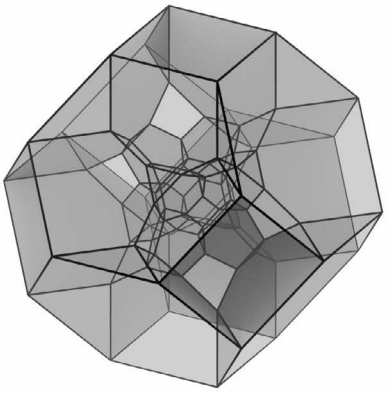
Hypercube



Truncated
hypercube



Rectified
hypercube



Bitruncated
hypercube
(perspective
projections)

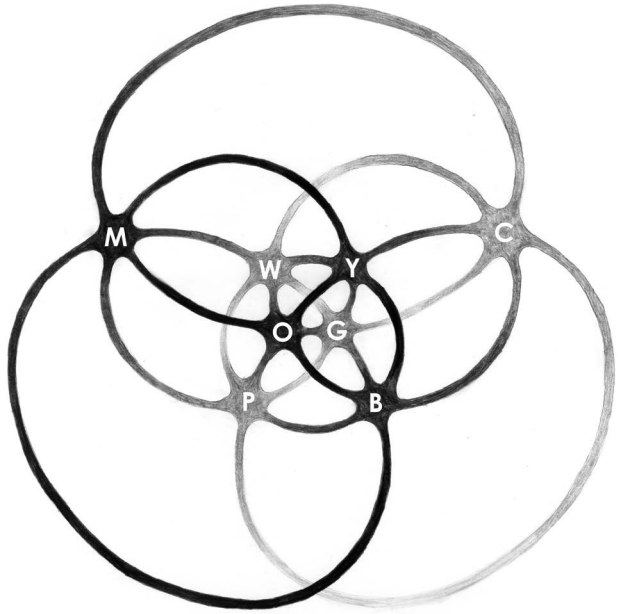


FIGURE 3.5: Connections of the eight truncated octahedra in the bitruncated tesseract: cyan (C), yellow (Y), white (W), magenta (M), black (B), green (G), orange (O), purple (P)

eight cells in the form of truncated octahedra (FIGURE 3.2, second from right) corresponding to the original cells of the tesseract, and sixteen cells in the form of a truncated tetrahedra (FIGURE 3.3, second from left and second from right) corresponding to the original vertices of the tesseract.

To arrive at a kinetic model, the trick is to implement the bitruncated tesseract *as a surface*. This is achieved by removing all the square and triangular faces of the polychoron, leaving us with a closed surface composed of sixty-four hexagons. As the hexagons meet four per vertex, the surface has negative curvature at these points.

THE PATCHWORK

Looking at the perspective projection of the bitruncated tesseract, it is evident that a kinetic model based on it should be able to change the size of its parts. This issue can be resolved by making the surface out of cloth, so the faces can wrinkle[160]. The hexagons

160 I used cotton because of its neutral aesthetics as a default mock-up fabric, through which to observe the geometric phenomenon without distractions.

FIGURE 3.6: The *Crooked House II* – A topological embedding of the bitruncated tesseract

are cut from cotton cloth of eight different colors, coming in pairs of (roughly) complementary colors. The complementary color pairs chosen here are cyan – orange, magenta – green, yellow – purple, and black – white[161]. Hexagons are sewn together first in groups of eight to create truncated octahedra, each of their own color. The cells are then connected by sewing together the edges around the removed square faces of the truncated octahedra according to the map given in FIGURE 3.5. The cells of complementary colors are not placed as neighbors, but as opposing cells of the structure.

To hide most of the seams on the inside of the model, I sewed the hexagons together from the other side of the surface, leaving one edge open. Then I turned the whole model inside out through that opening, and sealed it off from the frontside. The finished model—*Crooked House II*—is shown in FIGURE 3.6.

DETERMINING THE TOPOLOGY

The openings connecting the cells cause several handles in the patchwork surface. The exact number of handles can be verified by the Euler characteristic for surfaces V – E + F, for which we need the numbers of vertices (V), edges (E), and faces (F) of our surface. The number of the vertices must be the number of the hexagons (sixty-four) times the number of the corners in a single hexagon (six), all divided by the number of hexagons meeting at a vertex (four). Thus, the number of vertices in the patchwork equals ninety-six. The number of the edges must be the number of the hexagons (sixty-four) times the number of the sides in a single hexagon (six), all divided by the number of hexagons meeting on an edge (two). Thus, the number of edges, or seams, in the patchwork equals 192. These numbers give us an Euler characteristic of –32. For closed orientable surfaces such as our patchwork, the relation of Euler characteristic (χ) and the genus (g) of the surface is given by $\chi = 2 - 2g$, so the genus of the surface is seventeen. This means the patchwork surface is topologically a torus with seventeen handles.

161 The pastel hues are used for the same reason as in the previous model.

PLANES OF ROTATION

Let us first examine the rotations of a cube along the three perpendicular planes *xy*, *xz*, and *yz* (FIGURE 3.7). Notice that for a two-dimensional being living in the surface of the picture plane of FIGURE 3.7, only the rotation along the *xy* plane would look legitimate, i.e. rigid body movement. For the two-dimensional creature, the rotations along the *xz* and *yz* planes would seem like the figure is turning inside out. The square face originally on the exterior, enveloping the planar figure, is replaced with the neighboring one. From our experience with three-dimensional objects, we know that the squares appearing shrunken inside the enveloping square are actually sticking out of the picture plane into the third spatial dimension.

An analogous effect is witnessed in the patchwork model (FIGURE 3.8), as only the rotations along the *xy*, *xz*, and *yz* planes look like legitimate, rigid rotations. A quarter rotation along the *xw*, *yw*, or *zw* plane appears as a partial 'inside out turning' of the surface, where a cell neighboring the enveloping outer cell is pulled out through the square opening connecting the two, and the outer cell gets pushed inside through the opening on its opposite side. The bitruncated tesseract has now changed its orientation with respect to our 3D space, and the truncated octahedron in our hands has changed its color as a consequence. In reflection on the squished tetrahedron above, we know to interpret the cells stuffed inside the enveloping cell as sticking out of our space into the fourth spatial dimension.

CLOSING REMARKS

A person unacquainted with four-dimensional geometry might describe a 'crooked house' as some kind of a color changing pouch – a three-dimensional object interesting in its own right. In the context of four-dimensional geometry the model presents itself as a pedagogical tool offering a chance to explore the structure of the tesseract, and to demonstrate the rotations along the six perpendicular planes concurrent on the origin in Euclidean 4-space. It seems patchwork models are especially useful in visualizing surfaces living more naturally in spaces of higher dimensionality, as demonstrated by Elisabetta Matsumoto with her quilted model of the Klein quartic[162], a surface with fascinating mathematical properties.

162 Matsumoto 2017.

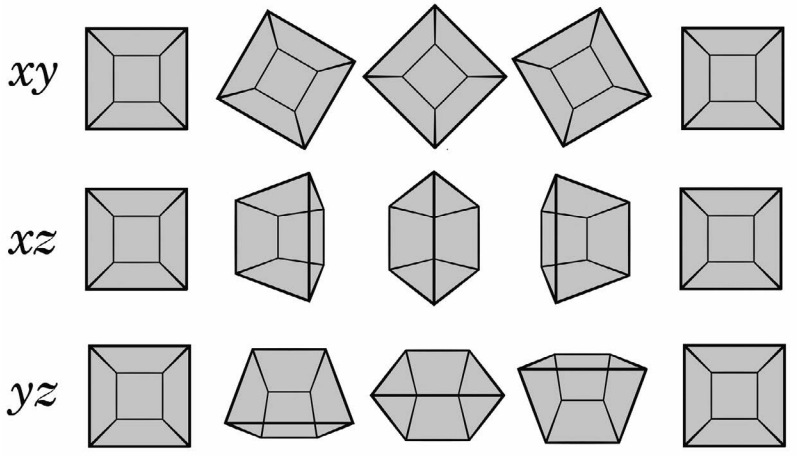


FIGURE 3.7: Quarter rotations of a cube along three perpendicular planes

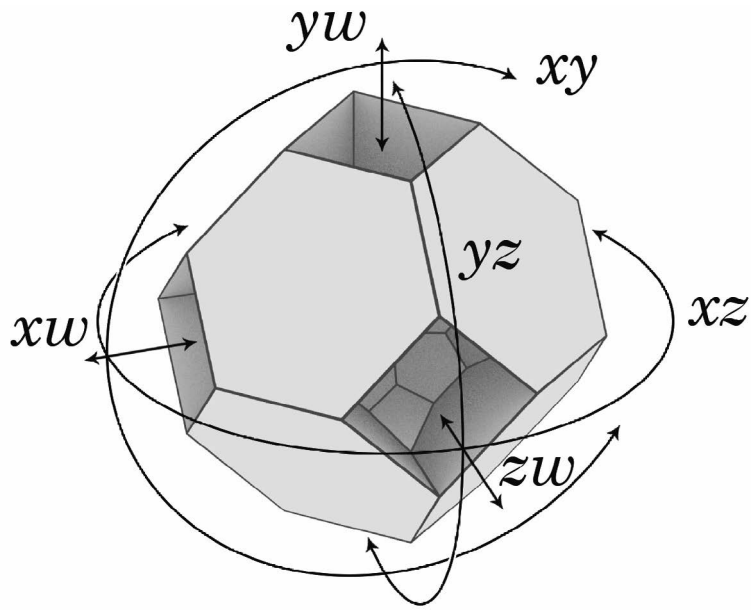


FIGURE 3.8: The appearance of rotations along six perpendicular planes in the patchwork model

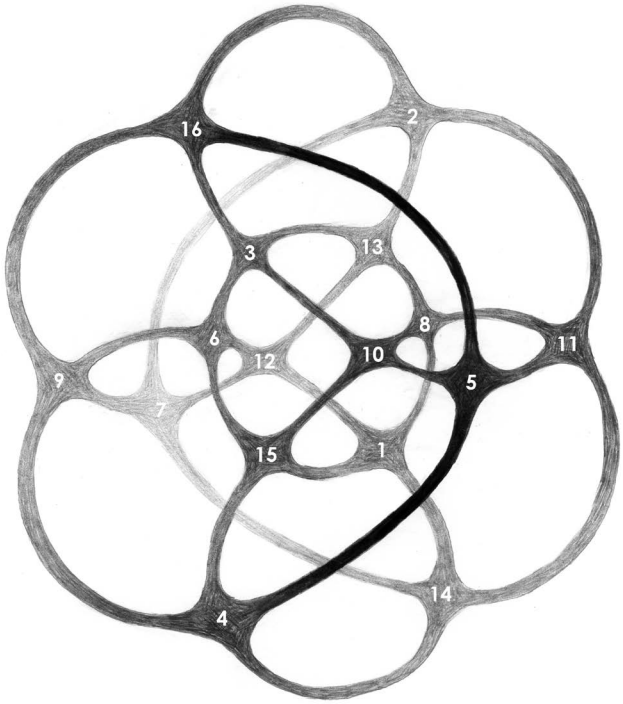


FIGURE 3.9: Connections of the sixteen truncated tetrahedra in the bitruncated hexade-cachoron

Besides the tesseract, we can examine the bitruncated forms of the other regular polychora to determine their suitability for patchwork visualization.

HEXADECACHORON

The truncation of a solid will eventually yield its dual solid, so the truncation of the tesseract will result in its dual, the hexadecachoron, also called the hexadecachoron. As the bitruncated form sits between the original solid and its dual, the bitruncated tesseract and the bitruncated hexadecachoron are actually the same thing. This means that the patchwork model of the hexadecachoron is exactly the same surface as described above, but viewed from the other side of the surface. The volume on that side consists of sixteen cells shaped like truncated tetrahedra, connected to each other via triangular openings (FIGURE 3.9).

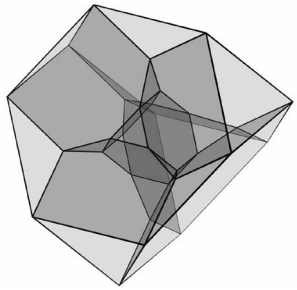


FIGURE 3.10: Bitruncated pentachoron (perspective projection)

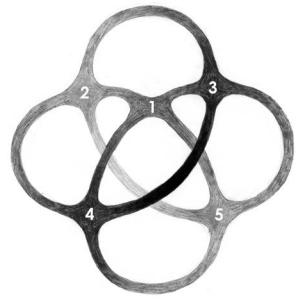


FIGURE 3.11: Connections of the five truncated tetrahedra in the bitruncated pentachoron

PENTACHORON

The bitruncation of the pentachoron results in a polychoron composed of ten truncated tetrahedra (FIGURE 3.10). When the triangular faces are removed, the remaining twenty hexagons form a closed surface. This surface has negative curvature at the vertices with four hexagons meeting, just like the surface discussed above. As its topology is significantly simpler – a torus with six handles – it is clearly a suitable subject for a simple patchwork implementation (FIGURE 3.12). FIGURE 3.11 shows how the cells of the bitruncated pentachoron are connected to each other via triangular openings.

ICOSITETRACHORON

The bitruncation of the icositetrachoron yields a polychoron composed of forty-eight truncated cubes (FIGURE 3.2, second from left). When the triangular faces are removed, the remaining 144 octagons form a closed surface. This surface is hyperbolic with four octagons meeting at a vertex. Its topological shape is relatively complex – a torus with seventy-three handles. It might not be feasible as a functional patchwork model, as pushing and pulling the abundance of cloth through the triangular openings might be too laborious.

HECATONICOSACHORON AND HEXACOSICHORON

The bitruncation of either the hecatonicosachoron or its dual the hexacosichoron yields a polychoron composed of 600 truncated tetrahedra and 120 truncated icosahedra, corresponding to the vertices of the hecatonicosachoron and the hexacosichoron, respectively. When the triangular and pentagonal faces are removed, the remaining 1200 hexagons form a closed surface. With four hexagons

FIGURE 3.12: The *Crooked House I* – A hyperbolic patchwork model of the bitruncated pentachoron

FIGURE 3.13: Crocheted version of the *Crooked House I* with constant negative curvature on each hexagon, by Kirsi Peltonen

meeting at the vertices, this surface again has negative curvature at these points. Its topology is exceptionally complex – it is a torus with 299 handles and is not feasible as a functional patchwork model.

Future improvements on the model would include careful consideration of a material – as seen in FIGURE 3.6, the cotton cloth gives a saggy and wrinkled appearance. Although more rigid material would articulate the cells more clearly, it might impair the movement of the surface through the openings.

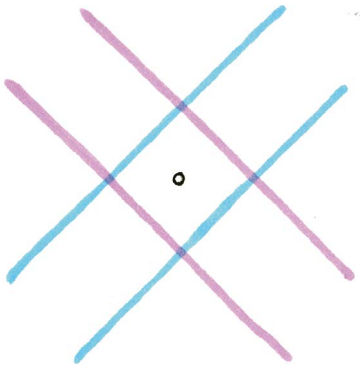
As for the actual geometry of the models, the polygons could also have curved edges to distribute the negative curvature along the edges instead of the vertices. Furthermore, if these polygons were cut from a stretchy material, the negative curvature would flow from the edges toward the polygons themselves.[163] To reflect the hyperbolic nature of the tiling with the highest fidelity, the patchwork can be made out of crocheted polygons instead of flat ones. Kirsi Peltonen, a senior lecturer of mathematics at Aalto University, implemented this idea in a physical model (FIGURE 3.13).

163 Weeks 2010.

MODEL 4: *Visit to the Ideal Plane*

In their 1936 survey into the art of mathematics, Hilbert and Cohn-Vossen described how a four-dimensional polytope called the icositetrachoron, when projected from its center into our three-dimensional space, yields an elegant figure called the Reye configuration. This idea is applied to a sequence of four models depicting a cube in a projective setting, and showing how the ideal plane, and the vanishing points on it, can be brought into view. The transition corresponds to the rotation of the icositetrachoron, and the subsequent change in appearance of its projections in three-dimensional space. This set of models was first presented in summer 2016 at the exhibition of the annual Bridges conference on mathematical connections in art, music, architecture, education, and culture.

a



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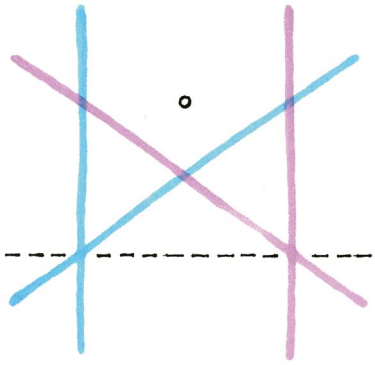

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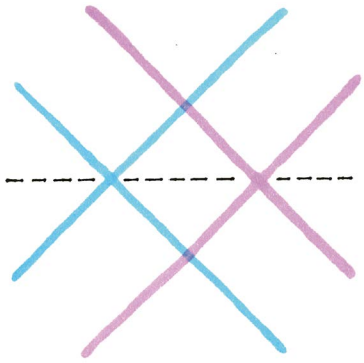


FIGURE 4.1:
Projective view
of a square

IDEAL ELEMENTS

René Magritte's 1955 painting "The Promenades of Euclid" shows a room with a canvas set up against a window, blocking it. The canvas contains the view outside – a spired tower top and a boulevard receding to infinity. The composition of this painting-in-a-painting is such that the side-by-side triangular images of both the conical spire and the boulevard are congruent. In addition, the apex of the spire hits the horizon line, just like the vanishing point of the boulevard. A banal illusion at first glance, the mention of Euclid in the title hints at a more challenging idea – the interchangeability of ordinary points and points at infinity.

The famous fifth postulate of Euclid states that given a line and a point not the line, there exists exactly one line through the point not meeting the original line. This line is said to be *parallel* to the first line. Mathematicians have later described how the so-called non-Euclidean geometries do not obey the parallel postulate. The most fundamental of these geometries is projective geometry, in which every pair of lines of the plane has an intersection: for parallel lines this intersection is thought of being at infinity, an ideal point. This idea was first experienced in perspective drawing, where the ideal points are called *vanishing points*, and in three-dimensional space they lie on an 'ideal plane'. Thus in the context of projective geometry parallel lines do not, strictly speaking, exist at all.

Let us take a square, and extend its edges to lines (FIGURE 4.1a), and color them so that parallel edges have the same color (blue and pink). We also mark the center of the square with a dot. Although seemingly parallel, in the projective context the edge lines intersect at two ideal points. This would be a confusing notion to a flatlander living on the plane of the square, but we can visualize these additional intersections easily in a three-dimensional setting by imagining ourselves hovering above the square, and then tilting our view, until the line at infinity, or the horizon, with the two vanishing points, appears to the view. At the stage shown in FIGURE 4.1b one of the vertices of the original square has traveled to the horizon. Its presence there is revealed by the fact that the lines originally meeting at that vertex have now become parallel. We can continue tilting our view until the original center of the square and two of its vertices have shot off to infinity (FIGURE 4.1c). At this stage the configuration again appears like a square, but the region of the original square is seen in two triangular parts, one above and another below the horizon line.

GNOMONIC PROJECTION

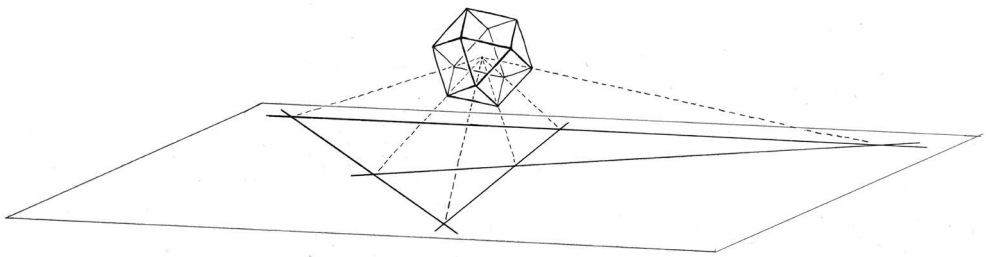
If we now examine this configuration of six points and four lines, we see that each point is an intersection of two lines and each line goes through three points. This regularity hints at another interpretation of this transformation – a special kind of projection of a higher-dimensional polytope. To produce the series of views in FIGURE 4.1 we can take a *gnomonic projection* of a semiregular solid called the cuboctahedron, and rotate it around its center.

The gnomonic projection looks at an n-dimensional object from its *center*. It has the advantage of mapping straight lines to straight lines, but unlike stereographic projection, it does not preserve angles. Gnomonic projection maps any two points on opposite sides of the center of the source structure to a single point in the image space. If the polytope portrayed is symmetric with respect to its center, the two halves of the structure are superimposed exactly in the image configuration. As such it can be interpreted as a depiction of only a half of the structure, now without any overlap of the elements.

FIGURE 4.2 shows the gnomonic projection of the cuboctahedron in a general position, where all the vertices have an image on the picture plane. It is a configuration of six points and four lines on the picture plane, where each point is an intersection of two lines, and each line is goes through three points. We can now interpret the ideal points in FIGURE 4.1 as being vertices of the cuboctahedron whose projection rays are aligned just parallel to the image plane, and are thus projected infinitely far away. For future reference, notice how the four lines of the configuration are gnomonic projections of cross-sectional hexagons of the cuboctahedron.

We will see how the idea of a configuration as a gnomonic projection can be applied to build a set of models of a *cube* undergoing a similar kind of transformation seen above with the square.

FIGURE 4.2:
Gnomonic
projection of a
cuboctahedron



a

b

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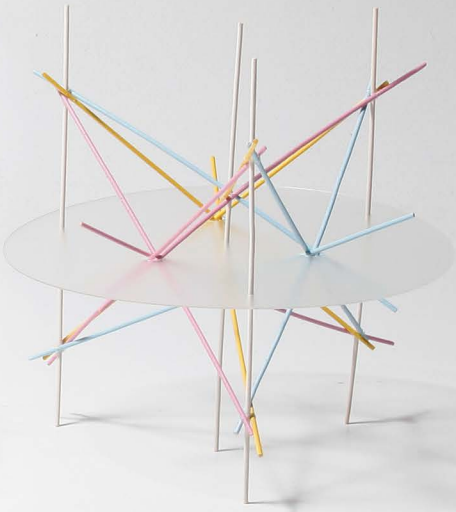

d

STICK MODELS

To start the sequence of models, we take a cube and extend its edges to lines (FIGURE 4.3a), and color them so that parallel edges have the same color (blue, pink, and yellow). We also mark the center of the cube with an intersection of white diagonals. Again, although seemingly parallel, in the projective sense the edge lines intersect at ideal points, and there are now *three* of them. As we did before with the square, a four-dimensional being could imagine itself observing the three-dimensional space containing this scene from a position 'above', and then tilting its view until the plane at infinity and the three vanishing points appear to the view. At the stage shown in FIGURE 4.3b one of the vertices of the original cube has traveled to infinity. Its presence there is revealed by the fact that the lines originally meeting at that vertex have now become parallel. We can continue tilting our view, and three more of its vertices have shot off to infinity (FIGURE 4.3c). At this stage the configuration again appears like a cube, but the region of the original square is seen in two triangular parts, one above and another below the ideal plane. In the final model, the three ideal points are positioned around the center of the model, and the center of the original cube has gone to infinity (FIGURE 4.3d).

THE ICOSITETRACHORON

Again, if we examine this configuration – twelve points and sixteen lines, we see that each point is an intersection of four lines and each line goes through three points. In the context of projective geometry this arrangement of points, lines and planes is called the Reye configuration, and shows up in many areas of mathematics. The regularity of the transformation can be interpreted as arising from it being a gnomonic projection of a higher-dimensional solid, this time a four-dimensional polytope called the icositetrachoron, having twenty-four vertices, ninety-six edges, ninety-six triangular faces, and twenty-four octahedral cells.[164]

By examining the planes determined by the rods, we see that there are twelve of them, and each of them is a gnomonic projection of a cuboctahedron as seen above. This is a natural consequence of the fact that the icositetrachoron has twelve cuboctahedral cross-sections, just as the cuboctahedron had four hexagonal cross-sections.

CLOSING REMARKS

The gnomonic projection is an unconventional method, and we are not used to interpreting its results spatially. Both the difficulty and the motivation to explore these projections come from the challenge of understanding a sequence of superficially different shapes as manifestations of the same higher-dimensional structure. Consequently it should not perhaps be the first choice to convey four-dimensional structures to an audience unfamiliar with the subject.

Interpreted as excluding half of the structure, gnomonic projection is a natural choice for portraying the more intricate members of the regular polychoron family. The loss of one half of the structure is not insurmountable, as most of the polychora are symmetric with respect to their center. When two antipodal vertices have their projection at a single image vertex, the imaginative effort required to infer the entire structure is not too taxing. Depending on the direction of the view, some elements of the structure might get projected at infinity, at which case their existence is revealed by the presence of parallel elements in the visible configuration.

The gnomonic projection supplements other methods by affording elegant views to local neighborhoods of the structures,

164 Hilbert and Cohn-Vossen 1956, 154–157.

and is well suited for portraying complex polychora with physical, reasonably-sized objects. The stick models can be built in reasonable time by students during workshops, and in different scales. They also present architectural possibilities, e.g. as designs for playground structures.

Gnomonic projections realized as objects differ from the usual visualizations in the sense that they are artificially clipped and understood as extending outwards to infinity. A curious feature of such interpretation is that a single cell can penetrate the plane at infinity and appear back in the picture from the opposite side, having vertices on seemingly disjoint regions of the configuration. This incompleteness gives them an interesting poetic quality, as the volumes defined by the rods are in the various stages of opening and reaching out toward the encompassing space, and establish a breathing, architectonic sense of interior.

In the gnomonic projections of many polytopes, the edges line up in pairs, triples, etc., with breaks in between. The pursuit of stick models leads one naturally to prefer the polytopes whose edges fall into continuous lines when observed from the center of the polytope. This feature has the advantage of making model building easier, as a single rod can be used to portray all the consecutive edges along it.

The polytopes fulfilling this requirement have the property that the vertex figure of each of them is a uniform polytope symmetric with respect to its center. In addition to the icositetrachoron, the polychora satisfying this condition are the hexadecachoron, the runcinated pentachoron, and the hexacosichoron. The gnomonic projections of hexadecachoron seems a bit too simple to have sufficient visual appeal, but the intricate weaving of seventy-two rods for the hexacosichoron is definitely worth an experiment in the future.

The runcinated pentachoron seems to guide the way deeper into the realm of projective geometry – a subject too vast to discuss further in the context of the work at hand. For example, it turns out the expanded version of the higher-dimensional simplices can be gnomonically projected to our three-dimensional space to obtain configurations embodying the generalizations of the celebrated Desargues' theorem for the perspectivities of simplices[165].

165 Luotoniemi 2018.

MODEL 5: *The Prismary*

The final model is an unconventional, 'oblique' depiction, which does not correspond to any legitimate projection. The polychora acting as the subject is not a regular one, but a prismatic structure lacking a precisely analogous solid in three dimensions. The model is an unstable skeletal mesh constructed from thin tubes and thread. As all of the edges are presented in their actual length, i.e. without foreshortening, the model could be called also a 'cavalier projection' – which is in fact not precisely a projection at all. Like the *Kinochoron* and *Crooked Houses* earlier, the model allows hyper-rotation through its open faces.

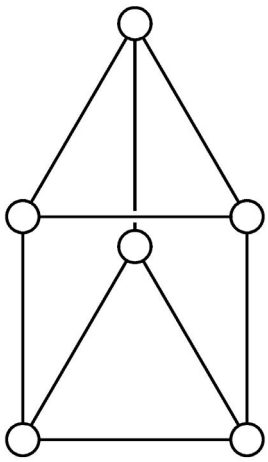

FIGURE 5.1
Illustration of the triangular prism

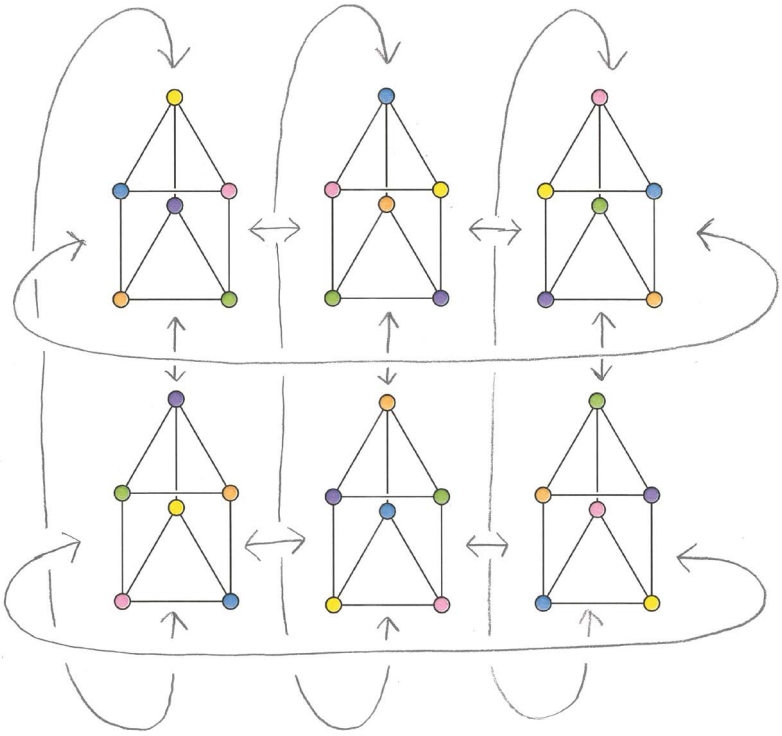

FIGURE 5.2
Map of the six possible orientations of the illustration

ROTATIONS OF A TRIANGULAR PRISM

Before introducing the actual model in question, let us make some preliminary observations. Although the 3-3 duoprism does not have a straightforward analog in the lower dimension, it is useful to pay attention to some features the *triangular prism* exhibits when depicted with a similar method we are going to employ here.

Note that if a triangular prism is portrayed so that all its nine edges have the same length (FIGURE 5.1), the contour of illustration consists of an undistorted triangle and a square meeting along an edge, as if unfolded down to the same plane. These two polygons enclose the other three, of which the triangle appears undistorted, and the two squares are distorted in a slanting manner, and they all meet at a vertex in the middle.

The three-dimensional interpretation of this planar illustration is that the undistorted triangle and square are in "front", and the other polygons are in the "back". When the triangular prism does a one-third rotation along the plane parallel to the pair of triangles, one of the square faces in the back appears in the front, and one square in the front disappears to the back. If the prism does a half turn around the plane parallel to a square, one of the triangular faces in the back appears in the front, and one triangle in the front disappears to the back.

FIGURE 5.2 shows the six different orientations the illustration can have, and how they are connected by one-third rotations (horizontal) and half turns (vertical) Notice that the graph-like structure of this map itself appears like a triangular prism, as the six possible states correspond to the six vertices of the solid.

THE 3–3 DUOPRISM

In three-dimensional space, the prisms have two polygons as their roof and floor, and a belt of quadrilaterals connecting their edges together. The natural extension of this concept to four dimensions is having two polyhedra as the "floor" and the "roof", and a spherical layer of three-dimensional prisms connecting their faces together. But in four-dimensional space there also exists another family of prismatic shapes, called duoprisms, which do not have an analog in three dimensions. The boundary of a duoprism consists of two sets of three-dimensional prisms, interlocked together like links in a chain.

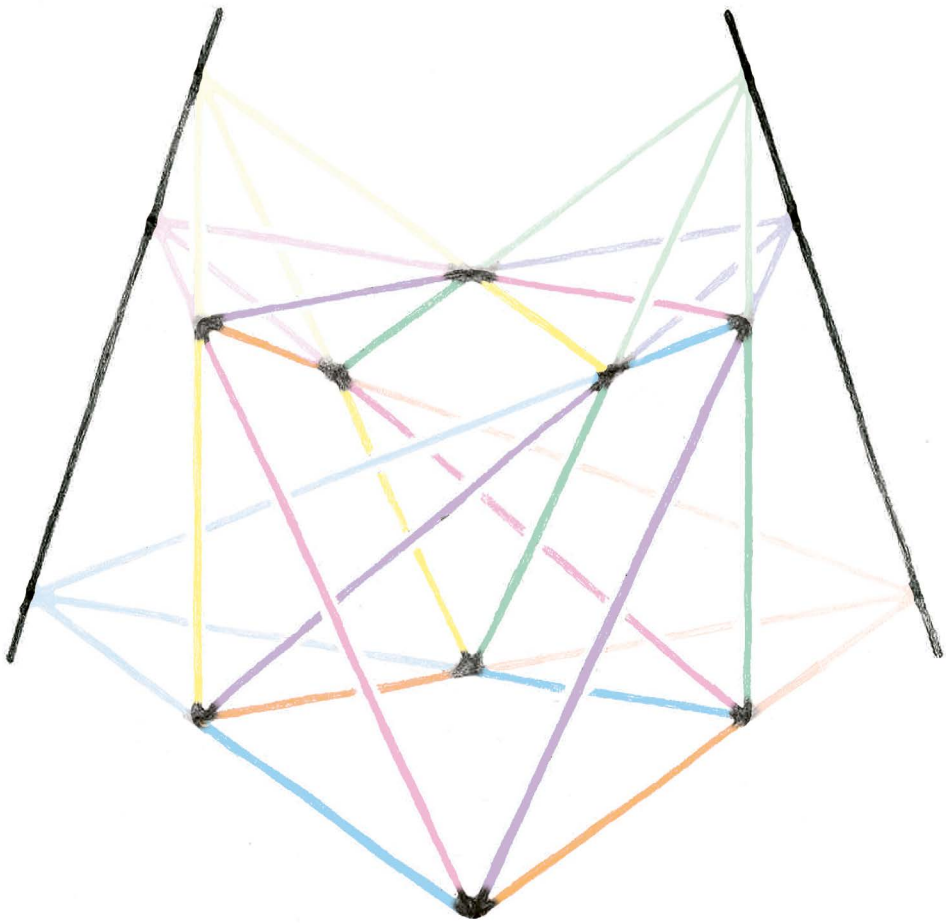

FIGURE 5.3: The 3-3 duoprism with its six vanishing points and two horizon lines

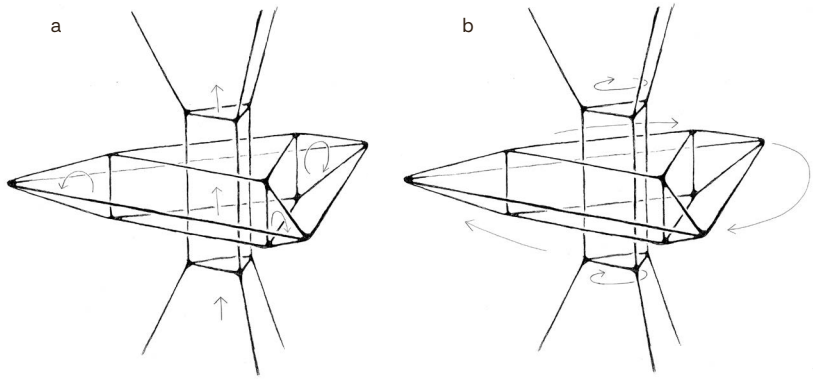

FIGURE 5.4: The chains of prisms rotating along and around their axis

The simplest of these polytopes is the 3-3 duoprism, which has nine vertices, eighteen edges, fifteen faces (six triangles and nine squares), and six cells. The cells have the shape of a triangular prism, and they come in two chains, where three prisms are joined at their triangular faces. These two chains are linked together, and are separated by the toroidal surface of 3×3 square tiling.[166]

FIGURE 5.3 shows how the parallel edges of the 3-3 duoprism (their extensions depicted with the lighter lines in the drawing) converge toward the vanishing points on the horizon lines, of which there are two – one for both of the chains of prisms.[167]

FIGURE 5.4a depicts each chain shrunk toward its own axis with a gap between them, to better illustrate the two interlocking rings. The view is centered on one of the triangular prisms, and the two other prisms in the same chain with the central prism are drawn only partially, blowing out toward infinity. This chain is rotating *along* its axis and the other *around* its axis. FIGURE 5.4b shows the complementary rotation that moves along the horizontal chain and around the vertical chain.

166 If six 3-spaces are thrown at random to a 4-space, one of the four-dimensional regions partitioned by the 3-spaces always has the shape of a 3-3 duoprism.

167 This configuration of fifteen points and twenty lines is related to the celebrated Desargues theorem, and the five-dimensional expanded simplex (Luotoniemi 2018).

KINETIC STRUCTURE

If a 3-3 duoprism is portrayed so that all its eighteen edges have the same length, the envelope of model consists of two undistorted triangular prisms meeting on a square face, as if unfolded down to the same 3-space. These two cells enclose the other four, which appears distorted in a slanting manner, and they all meet at a vertex in the middle. The four-dimensional interpretation of this model – as if seen by a four-dimensional observer – is that the two undistorted triangular prisms are in "front", and the other cells are in the "back".

The edges can be beaded together so that the ends of the straws touch each other in sets of four at the nine vertices. The result is a model – the *Prismary*[168] (FIGURE 5.5). In this model each of the six sets of three parallel edges have the same color, and thereby give a unique color to each of the six prisms. The colors are pastel hues of blue, magenta, and yellow on one chain, and orange, green, and purple on the other. The model is extremely simple in its construction; the only thing to take into consideration is to have some slack in the thread at the vertices to compensate the thickness of the tubes.

As the 3-3 duoprism can rotate in 4-space to reveal others of its facets, the model can be manipulated to interchange the positions of the prisms. This action is based on the fact that although the triangular faces are rigid, the square faces are unstable and can be folded and twisted to allow passage of other portions of the model through them. When the 3-3 duoprism does a one-third rotation along the plane parallel to a triple of triangles, one of the prismatic cells in the back appears in the front, and one cell in the front disappears to the back. In the physical model, these two prisms appear to turn inside out. We should keep in mind, though, that this eversion is only an illusory effect of the change in the direction they are viewed from in 4-space.[169]

FIGURE 5.6 shows the nine different stages the model can reach, and how they are connected by one-third rotations along

168 The name is a reflection on the rosary – a set of prayer beads. With this name I wish to connect the model to the mystic aspect of hyper-spatial history, and suggest a possibly meditative nature of the tactile hyper-rotations.

169 To visualize the kinetic action of the model it is perhaps useful for the reader to think about a three-dimensional triangular prism constructed with the same technique. It is easy to see that the prism can be turned completely inside out by twisting the model and pushing the two triangles through one of the squares one after the other.

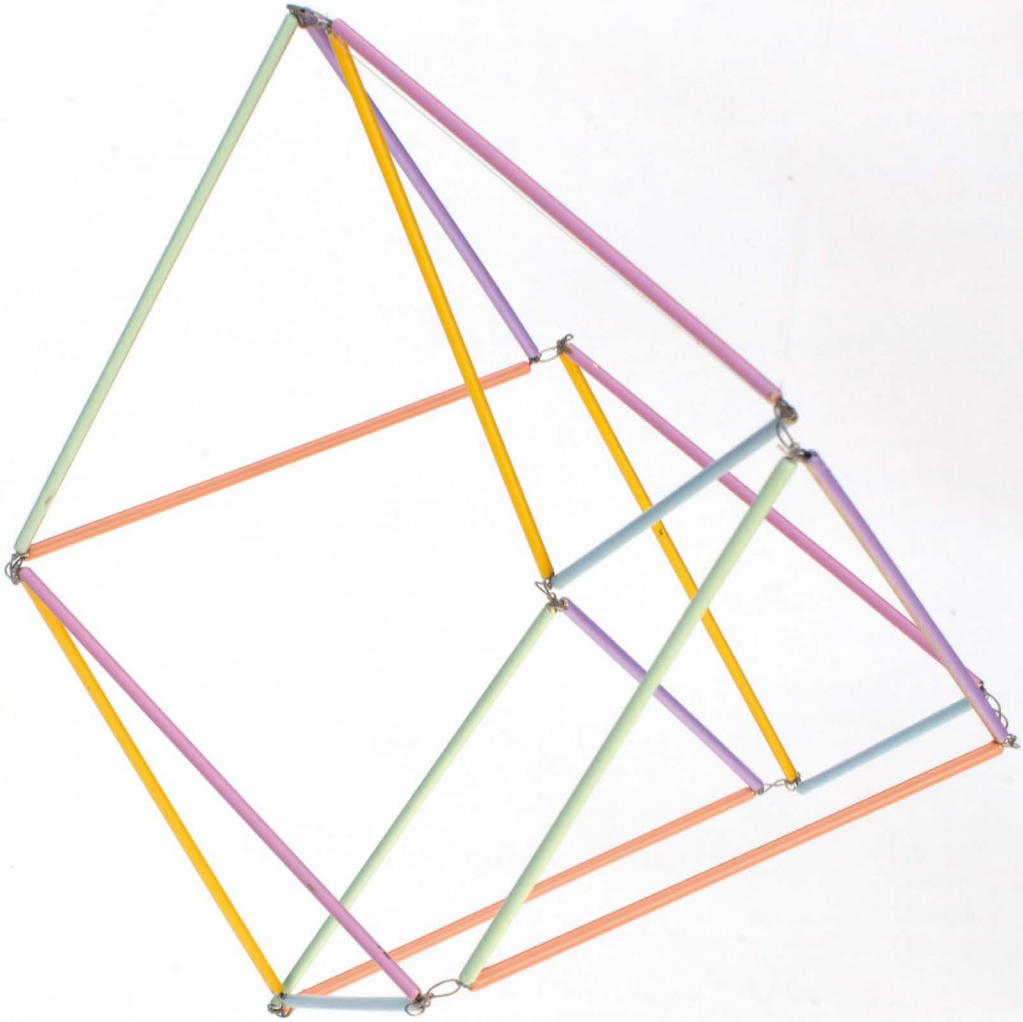

FIGURE 5.5: The *Prismary* – a kinetic model of the 3-3 duoprism

either of the chains. In the map the two chains are portrayed as horizontal (blue, magenta, yellow) and vertical (orange, green, purple). Notice that the graph-like structure of this map itself appears like a 3-3 duoprism, as the nine possible states correspond to the nine possible square-vertex pairs on the front and back sides of the four-dimensional solid, and the chains correspond to the triangular faces.

CLOSING REMARKS

As none of the edges of the *Prismary* are foreshortened, it does not correspond to any particular view of a regular 3-3 duoprism and would consequently look unconvincing to a four-dimensional observer. We can appreciate it as roughly resembling a parallel projection looking toward a vertex or a square face, and the metrical errors are acquitted by the kinetic action.

It seems that few of the polychora lend themselves to a kinetic edge skeleton like the 3-3 duoprism. It is evident that none the regular polychora composed of triangular faces – the pentachoron, the hexadecachoron, and the icositetrachoron – can be built with this method, and the rest of the regular structures constructed like this will collapse to a drooping, linear chain. In the less regular shapes, as the truncated polychora, the cells having triangular faces will not evert, and cannot consequently be rotated from front to back or vice versa. In light of these insights the 3-3 duoprism presents itself as having just the right combination of stable and unstable elements.

The *Prismary* makes a nice addition to the collection of models discussed previously, as it demonstrates a uniquely four-dimensional concept of duoprisms. The model also has a puzzle-like quality to it, as finding the correct set of manipulations for the hyper-rotation is not immediately evident.

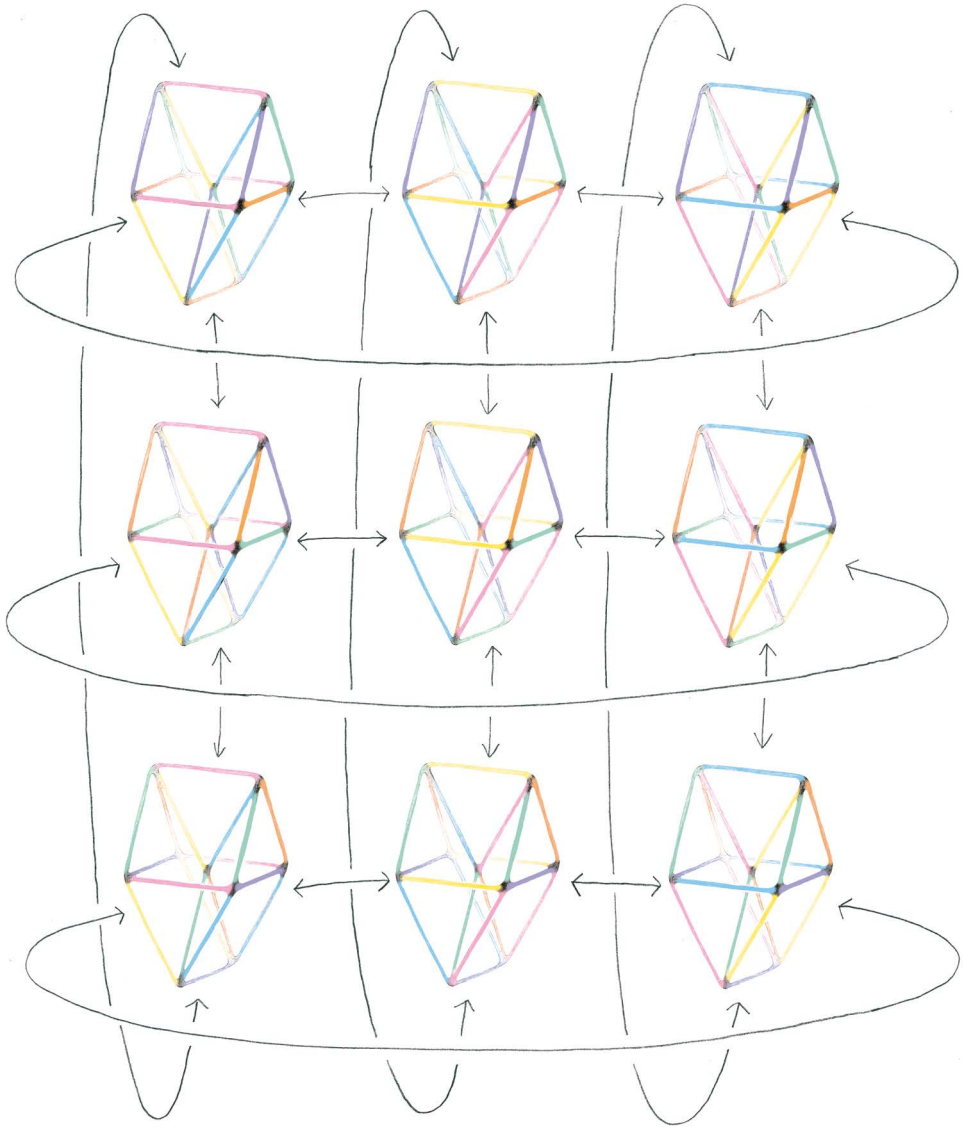

FIGURE 5.6: Map of the nine possible states of the *Prismary*

DISCUS-
SION

I will next move on to discuss and evaluate the investigation reported above and its results. We will see how this project situates in its cross-disciplinary context of mathematics and arts as a confidently visual artistic research, pedagogically motivated by the idea of inter-dimensional emancipation, and rich with poetic connotations.

MATHEMATICS AND ARTS (RE)INTEGRATION

The international community of scholars, teachers and artists who seek to integrate mathematics and art is vibrant and expanding. The phenomena studied in this interdisciplinary domain are easily understood and intriguing for the layman, and since the advent of powerful home computers have also become more available as an inspiration of recreational exercises. In particular, the Internet is riddled with pages where amateurs in the positive sense of the word enthusiastically discuss geometric and topological problems, often illustrated with computer-generated imagery. The millennia-long history of human interest toward forms, shapes, structures, and spaces has over the course of time been partitioned and reformulated as various fields of activity similarly to the other false dichotomies such as the 'emotional' and the 'rational'.

It often comes as a surprise to the general audience that mathematics is not just arithmetic, and geometry is not just measuring. As the discipline does not only strive toward giving explanations of reality, mathematics can sometimes resemble a form of art rather than a subcategory of science. As the investigation above reveals, the concept of the fourth spatial dimension in particular is situated in a peculiar way on the intersection of axes connecting particular art with universal mathematics, physical experience with abstract thought, and actual space with potential hyperspatiality, antipodes that at first glance seem far removed from each other. As such the concept makes an attractive subject for multidisciplinary methodology of artistic research and offers a first-class seat to perceive the interplay of the "mathematical" and the 'artistic' on problems that are not the private possession of either domain.

However, the disciplines differ in the manner they use to convince their audience of the newfound possibilities. In mathematics the assertions come in the form of proofs, the validity of which is governed by the most general principles of logical argumentation. In arts there is a corresponding notion, which Juha Varto calls *cogency*, which "is a result of the practitioner's discovery of a way of understanding a phenomenon, which is shared by many"[170].

170 Varto 2018, 60.

Although I lack the mathematical propensities, skills, and education to study and further develop the phenomena as pure mathematics, I see myself welcomed to work with phenomena that lend themselves to visual treatment, and often come under the categories of topology and geometry. Besides for its subject matter, the work is also sincerely mathematical in the sense that the focus is on the more general features of the cases. From my perspective, the topological procedure and structural insights that made the constructions possible are more important than the singular, one-of-a-kind objects themselves. An artist having a studio practice in some tradition of fine arts would no doubt have a stronger emphasis and a greater sensitivity toward the materials, media and production methods used. On the other hand, topological argumentation and its concentration on points, lines, and planes bears a remarkable resemblance to the close-to-practice, hands-on talk of an artist.

The results of my investigation here might look "mathematical" to an artist, and "artistic" to a mathematician. Being neither a professional mathematician nor an exhibiting artist, I cannot entertain high hopes of producing excellent results in either of these fields. Jyrki Siukonen, a sculptor and a scholar in arts, takes a cue from the fate of Emanuel Swedenborg and asks if it is plausible for an artistic research project to produce both significant art and significant research, either of these pursuits being quite demanding on its own[171]. A successful example, perhaps rare, is that of a doctoral research at Academy of Fine Arts of the University of the Arts Helsinki, where Markus Rissanen found generalizations of the famous Penrose tiling into new rotational symmetries – an actual mathematical discovery[172].

Usually however, the results of a cross-disciplinary research project probably suffer by comparison to those produced within an established field of study. In the same vein it can be concluded that the present work does not provide a serious contribution to either fine arts or pure mathematics, as it does not include any works of art, or theorems or proofs. The crafted models produced here do not function well if stripped from their "dual citizenship". Rather, any service my work here does to these disciplines is that it opens up surprising connections between them.

171 Siukonen 2002, 105.
172 Rissanen 2017.

RATIONALLY VISUAL ARTISTIC RESEARCH

Art is often described, especially by non-artist authors, as being a method of investigating the reality, the society, and the human condition[173], and research projects in art education especially are often targeted on "wicked problems" taken from humanities. There also appears to be a demand – maybe mostly from the scholars themselves, that out of artistic research should arise insights of a philosophical nature. The omnipresence of philosophy in arts is so indisputable that philosophically inclined research projects are usually not even declared as being cross-disciplinary. In such an environment, doing research concerned about points, lines, planes and spaces, and how they might look like from a specific viewing point, can feel a bit of a guilty pleasure. As seen in my investigation above, all considerations of a philosophical nature have escaped its scope. This has happened by necessity, as I am not in possession of the knowledge, skills, or the inclinations to make such a contribution, and can lay no claim to authority on these aspects of the subject. Instead, I have done the best service to my subject by applying my existing abilities toward the particularly visual treatment of hyperspatial geometry.

On the other hand, and as my example above demonstrates, the theoretical constructions employed within artistic research do not necessarily have to identify with humanities[174], and it is possible to consider other orientations. Although they might not fit well with our current taste or fashion, could, for example, the anatomical studies of Leonardo, the instructions on linear perspective given by Alberti, or M. C. Escher's hyperbolic tessellations done in collaboration with mathematician H. S. M. Coxeter be seen as early examples of artistic research? Without being a proponent of the Galilean view that "the book of nature is written in the language of mathematics", nor the Platonist school of philosophy of mathematics advocating the independent reality of mathematical objects, the results above permit the argument that geometric approaches pertaining to visual form can still be both relevant and effective in an artistic research project. Although geometric methods like the linear perspective and other projections might have lost some of their relevance in classic disciplines like sculpture, drawing and painting, they enjoy new

173 See e.g. Alva Noë's *Strange Tools – Art and Human Nature* (2016).
174 Nevanlinna 2002, 63.

attention in many digitally enhanced forms of visual culture, such as street art, projection mapping, computer graphics, game design, algorithmic architecture, virtual reality and 3D printing.

Another characteristic of artistic research has been an interest toward art practice for its concentration on the singular experience[175]. It has been seen as a counter-force for research attitudes in sciences, where the emphasis is on finding more general and objective knowledge of the subject matter. Consequently, in the context of academic discussions concerning artistic research, there is a concern often stated that the singularity of the process should not be lost. One practical consequence of these considerations is the problematization of academic prose – what kind of language should a research report use to reflect the nebulous nature of the art making itself? Although this may be a valid concern in research projects investigating the process itself, in an artistic research project such as described here, such fear seems futile. The assumed problem of losing the singularity of experience dissolves when we take the position that the purpose of the research is not to depict reality, but – to take a cue from John Dewey's pragmatism – to develop "means to regulate operations"[176]. The theoretical constructions put to use here are not meant to reflect the reality or be true to it, but are seen as tools to produce interesting visual effects. In other words, I am not interested in what four-dimensional geometry *is*, but what it can *do*, and the prosaic style of my research report above is justified through this position.

The aim of research work is often to "theorize", to translate sensuous experience into abstract concepts and frameworks. Here the situation is reversed – the abstract concept of the fourth spatial dimension is seeking to become embodied in a sensuous manifestation. Although geometric structures discussed in mathematics are perhaps intuitively thought of as being object-like, they are "objects" only in a metaphorical sense. Strictly speaking they are aggregates of locations or addresses – answers to the question "where?". The actual meaning of a concept of a point is accordingly *at that point*. Similarly the meaning of the concept of a line is *along that line*, and the meaning of a plane is *in that plane*. I tried to make those insights that are already mathematical rigorous to also become cogent in the context

175 Seen e.g. Vadén 2002.

176 Varto talks about the same practicality in connection with the principles guiding the practitioner's skill (2018, 55).

of intuitive, visual thinking. Democratizing pure mathematics through this intuitive access is also an important incentive for my pedagogical practice. It does not mean that the phenomenon will lose its rigor, as I hope my mathematically informed readers in particular noticed.

It is also important to emphasize that the investigation above is not about *perception*, a term which evokes the inclusion of human (or other organism) physiology. In visual arts there are many features that can be thought of as existing independent of human activity, e.g. those of shape, order, proportion, material, and other formal aspects of the practice. Those aspects of an object that involves only the object in itself, as if independent of any sentient observer, could perhaps call its *appearance*. On the other hand, the theories and practices in fine art that focus on the formal features are described as "modern", and therefore are often regarded by the contemporary art community as being old-fashioned. Maybe rather than labeling things as formalistic or modern, it would be more productive to argue what specific lines of investigation warrant an inclusion in the domain of artistic research.

In the context of this work, I have interpreted the "artistic" in artistic research in a rather naïve manner. I take it to refer to the fact that I have been mainly focused on the visual aspects of the phenomena. In most sciences shallow concentration on the visual would admittedly be a misleading, if not outright perilous approach. As a scholar in visual arts, however, I am authorized to occupy myself in investigation of visual matters that are superficial by definition, but are by no means trivial. The investigation detailed above is, however, set aside from the mathematical visualization, where the primary goal is to mediate the mathematics at hand rather than creating careful designs of visual appeal and interest. I hope my background in arts is also evident in the slight ambiguity of the "message", as I have wished to leave the artifacts open to several possible interpretations. They do not just have instrumental use as portrayals, but are supposed to give an experience of the "artistic", or perhaps even the "aesthetic", to resort to a philosophical vocabulary.

Often artistic research is conducted by artists who have a pre-existing artistic practice that they use as a research method. Although the key components of this research project – the subject matter, theoretical framework, methods, materials, and cultural references – are selected in an entirely "authoritarian" manner (i.e. in

the manner of an author)[177], the research methodology used here differs from fine art practice in several respects. As I cannot lay claim to such skills, I argue my lecturing work – with the preparation of the materials such as detailed above included therein – to constitute an artistic practice.

I am sincerely interested in the general geometric and topological principles my results exhibit, and consequently do not introduce them as "art works" with an emphasized particularity. As pedagogical tools, the models are meant to be copied, and developed further. To reflect the standard ideal of repeatability in the scientific method, perhaps it is the building instructions which should be considered as the actual results of the research, not the particular, singular models built by myself. The challenges involved in the development of the models into finished art pieces through careful consideration of materials, fabrication techniques, scale, etc. have not been the focus of this study.

I should note also that my artifacts are not "self-contained" in the same sense as art works – they require explanations in the form of e.g. accompanying documentation or lectures. Siukonen uses the 1974 writings of Finnish painter Juhana Blomstedt as an example to show how the rationality of the documentation is sometimes even a desperate solution[178]. The research report above cannot be accused of the same sin, as the process really was rational – at least when presented as such. The models above could not have been reached instinctively because the hyperspatial struggles required were mainly of a cerebral nature. They are not "artist's conceptions": instead I have tried to clearly state what is the relation of the model and the actual phenomenon represented, and answering this demand has taken up most of the technical report of the thesis.

INTERDIMENSIONAL EMANCIPATION

At Aalto University, I have had the privilege to lecture on geometric and topological topics to art, design, and engineering students in the context of various interdisciplinary courses on mathematics and art. During these sessions I try to convince my audience that space is not a passive backdrop in visual practice, but that it resists and shapes all efforts through its mathematically described properties,

177 Varto 2018, 87.
178 Siukonen 2002, 48.

such as curvature, orientability, genus, and, as demonstrated in this study, dimensionality. I have often used the models described above (or their prototypes) to demonstrate hyperspatial phenomena. In their feedback the students have repeatedly stated that the challenging concepts and structures of four-dimensional space become far easier to grasp through the use of these physical and kinetic visualizations. A more careful investigation into the instructive usefulness and impact of the models remains a natural topic for a follow-up study.

Although four-dimensional geometry is arguably a challenging concept for most of us, I argue that the main difficulty does not lie in the complexity of the subject. Rather, there is the preliminary step of being willing to participate in a thought experiment, reside for a while in lofty heights of abstraction, and not hasten to judge the concept by its correspondence or applicability into the real world. This of course requires some imagination on the part of the student. It has been my experience that audiences with a background either in mathematics or in arts are especially apt to give serious consideration to such counter-intuitive thoughts and ideas.

As four-dimensional geometry has had a lasting effect on Western culture and is included in the contemporary scientific worldview, it is, however, a suitable subject of interest for any audience. Higher-dimensional thinking can also be applied to reach a more complete understanding of phenomena other than space, such as many-dimensional data structures in any quantitative research discipline, or more famously, the passing of time. In 1866 Clifford was already using the idea of higher space to formulate geometrically and solve a probability problem regarding a line broken up to pieces and assembled into a polygon[179], and Hinton demonstrated the utility of four-dimensional thought with an example of a set of swords having the qualities of brightness, length, sharpness, and weight[180]. The treatment of such dimensional problems certainly benefits greatly from the prospect of also being able to grasp them visually.

Interdimensional thinking helps us to observe situations of some particular dimensionality as occurrences more of a general phenomenon, and also affords us to see the spaces of lower dimensions in a new light. We realize that the effect manifested by simple two-dimensional projections – the emergence of measurably

179 Clifford 1959.
180 Hinton 1901, 4–5.

different configurations as appearances of a single, higher-dimensional structure – would be a preposterous notion to a two-dimensional creature existing and acting exclusively along the picture plane. In fact, it would be extremely difficult to convince a flatlander that such figures are actually portrayals of the same object. The imaginative leap of faith we demand the flatlander to take is kept in the backs of our heads as we climb to the next rung of the dimensional ladder.

Understanding the preconditions and limitations laid down by dimensional space is important for artists seeking to master their media. These formal characteristics can often be described with concepts existing in geometry and topology. The significance of an art practice informed of higher space arises from its potency to enrich the formal vocabulary of three-dimensional shapes with novel means and meanings. After an interdimensional detour into hyperspace we may return to the safety of our 3-space enhanced by, for example, a capability to interpret a sculptural form in the same way as a picture – as a necessarily flawed portrayal of a higher-dimensional subject. In this sense the current work could be seen as science education for artists rather than art pedagogy.

My choice of building crafted models with the traditional approach of descriptive geometry also supported the pedagogic ambitions I had. Even risking the accusation of some kind of Thoreauvian-Luddite romantic nostalgia, I have chosen to interpret my digitally sketched models in classic materials and techniques. This kind of attitude is sometimes referred to as *post-digital*, although the concept has several other meanings as well[181]. A computer scientist might consider my procedures cumbersome, as projections can usually be produced instantly from a list of vertex coordinates analytically with linear algebra, as in Michiels' python script, which I used in *Cube at Large* to experiment with the sculptural composition of the model. I argue that the synthetic construction methods and argumentation I used in my visualization projects serves to better illustrate the various projections and the appearances acquired through them.

Notwithstanding the above, I have to admit that the projects I chose as cases for this dissertation are not the ideal for a layperson coming into contact with four-dimensional geometry for the first time. During the courses I teach at my home university, my

181 Cramer 2014.

students spend several preparatory lectures and workshops learning about topology, projections, and polytopes in various dimensions before getting exposed to the artifacts described here. Indeed, my models are not perhaps the most relevant, efficient, or appropriate early introductions to the topic. The best primer to 4-space – or at least to its geometry – is arguably the selection of interactive computer software. The reason I tried to find different routes was because I held the novelty of the projects and the models resulting from them in high regard. As this is an original academic work, I strove toward designing visualizations not seen before.

POETIC CONSIDERATIONS

The research project reported here unfolded as a focused concentration on the appearance of *things that do not exist*, or at least that have a purely mathematical existence. During the work, the delightful uselessness of my endeavor brought me great pleasure[182]. Consequently, if the fourth dimension of space were ever found in physics, the value of the present study in my own eyes would perhaps rather decrease than increase. Indeed, I was fascinated in the topic precisely as a thought experiment, and how far it could be developed while preserving its logical rigor. We saw how the careful analysis and description of the phenomenon can isolate its quantitative elements and thus allow them to be incremented in dimension. The new incremented formulation thus takes on an almost poetic mode, where nouns are joined to verbs to perform functions that seem initially nonsensical. Within four-dimensional geometry we are confronted by notions that at first hand seem to be built from incompatible pieces by an unchecked poetic fancy, if not outright absurd. How can a space "curve", how can volumes "bound", how can something rotate "around a plane"?

On this side the connection of mathematics and art is particularly solid – both activities are heavily dependent on the skill of imagination. There are practitioners of both disciplines who are not necessarily concerned about the actual existence of their phenomena in the observed reality, but instead the subject matter is often "such stuff as dreams are made on" As such, the results of both mathematics

182 Similar thoughts on the usefulness of one's own work are described by G. H. Hardy in his 1940 *A Mathematician's Apology* (Hardy 1967, 150).

and art cannot be falsified by comparison with the physical world, nor do they ever expire. Instead, the mathematical results can be falsified by comparison with the "mathematical world" with e.g. a counterexample, and a piece of art might not "work".

One consequence of this sort of mathematical or artistic activity is the awakening of a sense of possibility, to which actual reality manifests itself as just one possible world among many others. In an art practice, such as science fiction, it is extremely difficult designing these worlds to be truly novel. The imagined realities always seem to end up repeating the current societal and technological concerns in a disguise. Mathematics offers us if not a promise, at least a prospect of circumnavigating the human, or perhaps even the animal viewpoint. Although we know from cognitive psychology that even the most abstract mathematical concepts are built metaphorically from sensory-motor experience[183], it is evident that mathematical effort brings forth counter-intuitive and provoking ideas that otherwise would not have unveiled themselves. The strength of mathematics is that through its analytic and numerical practices, it can arrive at conceptions of shapes and spaces that are genuinely alien to everyday experience, but still rigorous and logically consistent. As such they present us with a singular breed of fiction transcending the human imagination.

Mathematicians and artists both share the idea that there is more than meets the eye. Hyperspatial geometry talks to us through a poetic mode by establishing connotations outside the realm of rational, or even real. Four-dimensional space can be perhaps poetically seen as a domain like the dreamtime of Aboriginal Australians, a more real reality, a place that can be accessed only by shamans or through psychedelic practices. But like Plato's ideal realm, geometric hyperspace is a place that is conceived through logical and analytical methods. What makes four-dimensional space a contradictory notion at its face value is how it manifests itself simultaneously as a completely rational yet literally supernatural concept.

Although the argumentation in the research report above is purely rational in its geometric development, the esoteric history of hyperspace gives it a subtext – another, more suggestive mode creeping in between the lines, an eerie understanding of the

183 Lakoff and Nuñez 2000.

hidden reality looming large behind the docile façade of points, lines, and planes. Indeed, through four-dimensional geometry we can experience a kind of dimensional horror[184] already employed in the hyperspatial passages of H. P. Lovecraft's ominous *The Dreams in the Witch-House* (1932). There is even a newspaper report of an Oxford undergraduate who committed suicide on the spring of 1900 after an intensive period of studying the connection of religion and science through the fourth spatial dimension.[185]

When Gardner featured Hinton's cubes in his *Scientific American* article, he received a grave warning from an English consulting engineer who had first-hand experience with the method. The letter claimed that the four-dimensional visualization exercises took on a life of their own, and eventually the sequence of cubes would "begin to parade themselves through one's mind of their own accord". He went on to declare the exercises "completely mind-destroying" and that he would not "recommend anyone to play with the cubes at all".[186] In a more humorous vein, we saw how in the case of the *Crooked Houses* above the polyhedra could be tucked away as in the "hammerspace" of animated cartoons, and made to emerge again like clowns from a car.

It is fascinating to think how the models could become inhabited, or even possessed, by their hyperspatial content through a "sympathetic magic" akin to that of James George Frazer. Each of the crafted models above can be seen set up like a Jacob's ladder, a two-way connection between our plane of existence and a morphological land-of-plenty from where shapes appear in a proper "deus ex machina" fashion. The kinetic models particularly can be thought of as prayer beads, as silent incantations. Thus they would act as signals – like the castaway's s.o.s. scrawled on the sandy plane of the desert island, calling out to hyperspace to "save our souls" from the three-dimensional claustrophobia. Or perhaps these objects are messages to the denizens of hyperspace, pointing toward them saying: "We're onto you".

184 Mark Blacklock makes a connection of this aspect of hyperspace to agoraphobia (2018, 197).
185 Blacklock 2018, 194.
186 Gardner 1978, 52.

IN CONCLUSION

The peculiar character of mathematics is that it can make outrageous claims, and then, rather surprisingly, make us believe them using convincing argumentation. It is through this mechanism that mathematics can challenge and enhance our perception of shapes and spaces. For example, topology tells us that a coffee mug and a donut are actually the same shape, and in the context of projective geometry the vanishing points – objects as illusory as the end of the rainbow, can be seen as concrete as the corners of a cube, and even interchangeable with them. With the work above I have wanted to share the experience of bewilderment I myself felt when I first came into contact with the similarly preposterous idea of four-dimensional space. It continues to surprise me how such an outlandish idea can be developed into logically convincing concepts.

In the beginning of this artistic research investigation, I set out to design new kinds of three-dimensional models of the four-dimensional regular polychora – mathematical hyperobjects residing in the fictitious four-dimensional space. Although making sketches with 3D-modeling software, I crafted the final objects from traditional materials with classic fabrication techniques because visualizations implemented in digital media are already well presented in the field, and because I wanted to strengthen the pedagogical value of the treatment. As a result we saw the appearance of five visualization cases exhibiting curious three-dimensional effects that guide the way toward various interdimensional insights and hyperspatial interpretations. The *Kinochoron* exhibited the peculiar behavior of seemingly simple interlacing of wire hoops, and in the *Cube at Large* we saw the strangely consistent coexistence of simultaneous perspectives, topped off with the optical illusion observed at the vanishing points. The *Crooked Houses* manifested startling sequences of eversions and color changes of the cloth pouches. An unexpected effect was also how the three-dimensional regions in the *Visit to the Ideal Plane* unfolded toward and reappeared from infinity, and finally in the *Prismary* we witnessed surprising permutations of the beaded structure.

To an audience initially ignorant of the geometry behind the concept, the speculations regarding four-dimensional space might seem arbitrary and unrestricted. However, they belong to an exceptionally precise line of theoretical study that also offers a fair share of disappointments and surprises to a researcher. The reason

why four-dimensional geometry counts as research is that it *resists*[187] research investigations through its prerequisite for logical rigor. Within the research project detailed above we saw how the fourth dimension also resists our attempts to visualize it. This opposition was also one of the main motivations for the work, and the investigation justifies itself as it made the models to appear.

The making of the artifacts can be seen, in a sense, as a 'close reading' of four-dimensional geometry. Their beauty is that of an *affordance*[188], as they provide us with the possibility of experiencing provoking spatial revelations and insights. Indeed, it is unlikely that anybody would ever have stumbled onto designing such three-dimensional artifacts without specifically hyperspatial aspirations. Through its exposure to hyperspatial influence, our ordinary 3-space is subjected to a metamorphosis at the location an "interdimensionally enhanced" object occupies, and it is as such how the models enter into the "life of forms" – to take a cue from art historian Henri Focillon[189].

The year 2018 saw the publication of three individual monographs on the history of the fourth spatial dimension. A doctoral dissertation in literature, Mark Blacklock's *The Emergence of the Fourth Dimension*, establishes the fin-de-siècle cultural influence of hyperspace, Klaus Volkert deals with the idea from the viewpoint of the history of mathematics in *In höheren Räumen: Der Weg der Geometrie in die vierte Dimension* with an emphasis especially on the German developments of the subject, and Cristopher G. White's *Other Worlds: Spirituality and the Search for Invisible Dimensions* addresses the subject in the context of religious studies. Although the cultural history of hyperspace was not the topic of my investigation here, the poetic aspects of my subject matter seen through the references in art, occultism, fiction, etc., greatly motivated the building of the models – an inspiration I also strive to share through my lectures and demonstrations.

187 Varto refers to Leibniz's concept of resistance as a criterion of something being "real" (2018, 89).

188 In the sense of James J. Gibson's perception psychology.

189 Focillon 1989.

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IMAGE SOURCES

FIGURE 0.5: Stringham 1880, 15.

FIGURE 0.6: *Projection of the four-dimensional 24-cell* (http://modellsammlung.uni-goettingen.de/).

FIGURE 0.7: Hinton 1901, frontispiece.

FIGURE 0.8: Jouffret 1903, 153.

FIGURE 0.9: Coxeter 1973, plate VIII.

HYPERSPACIAL INTERLACE

Four-dimensional space is a mathematical thought-experiment involving adding an extra spatial dimension perpendicular to our three dimensions of length, height and width. Research on the properties of hyperspace is made possible by generalizing the methods acquired by studying more familiar spaces of lower dimensions. Originating in philosophy and mathematically formulated in geometry, the concept has inspired interpretations in mysticism, in theoretical physics, in fiction and in the visual arts.

Just as three-dimensional objects can be drawn, unfolded, sliced, photographed or otherwise portrayed onto a planar medium, these graphical techniques can be generalized to produce three-dimensional appearances of the 4D structures described by mathematicians. Hyperspatial Interlace – a doctoral work in the interdisciplinary context of mathematics and art, studies new possibilities for visualizing hyperspatial geometry.

Hyperspatial reasoning offers artistic research a provokingly counter-intuitive, but nevertheless logically consistent framework rich with scientific, historical and poetic significance. The sensuous accessibility provided by physical artifacts and the simple vocabulary of geometry makes the research easy to share across various disciplines.

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